10/1 Clustering Data (L1)

1.1 Problem Statement:

Let $\mathcal{X}$ be a metric space. $d: \mathcal{X} \times \mathcal{X} \mapsto \mathbb{R}^+$

$x^{(i)} \in \mathcal{X}$, $i \in Z = \{1, \dots, N\}$

Also then $x^{(\cdot)}: Z \mapsto \mathcal{X}$

Problem: Divide the data set $\{x^{(i)}\}_{i \in Z}$ into k clusters.

Many methods: PCA, K-means, Graph Laplacian...

1.2 Graph Laplacian:

$G = (Z, E)$ ~ Z: vertices/nodes, E: edges

$E = \{(i,j) \mid i \in Z, j \in Z\}$

$\phi: Z \mapsto \mathbb{R}$ ~ $\phi \in \mathbb{R}^N$

$A: E \mapsto \mathbb{R}$ ~ $A \in \mathbb{R}^{N \times N}$

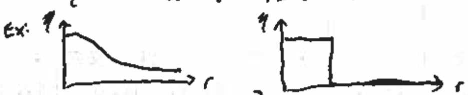
Definition: $\eta: \mathbb{R}^+ \mapsto \cancel{\mathbb{R}}\,[0,1]$ is a weight function if:

(i) $\eta(0) = 1$

(ii) $\lim_{r \to \infty} \eta(r) = 0$

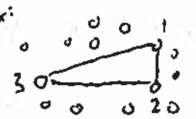
(iii) $\eta \downarrow$ i.e. $\eta(r_1) \le \eta(r_2)$ if $r_1 \ge r_2$

Ex:



Definition: $W: E \mapsto [0,1]$ is the weighted adjacency matrix defined by $W_{ij} = \eta(d(x^{(i)}, x^{(j)}))$

Ex:



$0 \in Z$
$X = \mathbb{R}^2$
$d(x,y) = |x-y|$

$W_{12} \ge W_{23} \ge W_{31}$, W is symmetric

Ex: why is this useful?

$x \in \mathbb{R}^d$, $d \gg N$ we replace a $d \times N$ matrix with an $N \times N$ one.

Definition: The weighted degree matrix $D: E \to \mathbb{R}^+$ is

$$D_{i,j} = \delta_{ij} \sum_{k \in Z} W_{ik} \quad (\delta \text{ is kronecker delta})$$

i.e. D is a diagonal matrix with diagonal vector $d: Z \mapsto \mathbb{R}^+$ defined by $d_i = \sum_{k \in Z} W_{ik}$

Definition: The (unnormalized) graph Laplacian $L: E \mapsto \mathbb{R}$ is defined by $L = D - W$

Theorem: a) $\mathbb{1} = (1, 1, \dots, 1)^T \in \mathbb{R}^N$, then $L\mathbb{1} = 0$

b) L is symmetric and positive semidefinite

c) If $W_{ij} > 0\ \forall (i,j) \in E$, then $\mathbb{1}$ is the only eigenvector with eigenvalue 0

Proof: a) $(L\mathbb{1})_i = \sum_{j=1}^N L_{ij} = \sum_{j=1}^N D_{ij} - \sum_{j=1}^N W_{ij} = D_{ii} - \sum_{j=1}^N W_{ij}$
$= \sum_{k=1}^N W_{ik} - \sum_{j=1}^N W_{ij} = 0$

b) $d(x,y) = d(y,x)$ (property of metrics) ~ $W^T = W$

Euclidian inner product $\langle \cdot \rangle$

$\langle \phi, L\phi \rangle \ge 0\ \forall \phi \in \mathbb{R}^N$ (need to show this)

$\langle \phi, L\phi \rangle = \sum_{(i,j) \in E} \phi_i L_{ij} \phi_j = \sum_{(i,j) \in E} \phi_i (D_{ij} - W_{ij}) \phi_j$

$= \sum_{i \in Z} \phi_i^2 D_{ii} - \sum_{(i,j) \in E} \phi_i W_{ij} \phi_j = \sum_{(i,k) \in E} \phi_i^2 W_{ik} - \sum_{(i,j) \in E} \phi_i W_{ij} \phi_j$

$= \frac{1}{2} \sum_{(i,k) \in E} \phi_i^2 W_{ik} + \frac{1}{2} \sum_{(j,k) \in E} \phi_j^2 W_{jk} - \sum_{(i,j) \in E} \phi_i W_{ij} \phi_j$

$= \frac{1}{2} \sum_{(i,j) \in E} \phi_i^2 W_{ij} + \frac{1}{2} \sum_{(i,j) \in E} \phi_j^2 W_{ji} -$ —//—

$= \sum_{(i,j) \in E} \underbrace{\tfrac{1}{2} W_{ij}}_{\ge 0} \underbrace{|\phi_i - \phi_j|^2}_{\ge 0} \ge 0$ (keeping in mind $W_{ij} = W_{ji}$)

c) All eigenvectors are orthogonal. Let ϕ be any eigenvector not proportional to $\mathbb{1}$. Then $\phi \perp \mathbb{1}$. Furthermore then $\exists (l,m) \in E: \phi_l \ne \phi_m$.

$\langle \phi, L\phi \rangle \ge \underbrace{\tfrac{1}{2} W_{lm}}_{>0} \underbrace{|\phi_l - \phi_m|^2}_{>0,\ \phi_l \ne \phi_m}$ by b)

$\therefore \langle \phi, L\phi \rangle > 0$; $L\phi = \lambda\phi$; $\lambda \langle \phi, \phi \rangle > 0$; $\lambda |\phi|^2 > 0$; $\lambda > 0$

Ex: $X = \mathbb{R}^d$; $d(x,y) = |\cdot|$; $\eta(x) = \exp\left(\frac{-x^2}{2\delta}\right)$; $\therefore L = L(\delta)$

1.3 Eigenvalue Problem: (EVP)

$(*)$ $\left.\begin{array}{l} L\phi = \lambda\phi \\ |\phi|^2 = 1 \end{array}\right\}$ Find $z = \binom{\phi}{\lambda} \in \mathbb{R}^{N+1}$ so that $(*)$ holds.

$F: \mathbb{R}^{N+1} \mapsto \mathbb{R}^{N+1}$ by $F(z) = \begin{cases} L\phi - \lambda\phi \\ \langle \phi, \phi \rangle - 1 \end{cases}$ ~ F is nonlinear

The EVP is to find 0's of F

$L(\delta)\phi = \lambda\phi$; $|\phi|^2 = 1$ $\quad F(z, \delta) = \begin{cases} L(\delta)\phi - \lambda\phi \\ \langle \phi, \phi \rangle - 1 \end{cases}$

$F: \mathbb{R}^{N+1} \times \mathbb{R} \mapsto \mathbb{R}^{N+1}$

Now solution $z = z(\delta)$; $z: I \mapsto \mathbb{R}^{N+1}$, $I \subset \mathbb{R}^+$ interval

$z \in C(I; \mathbb{R}^{N+1})$ (Implicit Function Theorem)

10/3 L3: Partial Differential Equations

3.1 Functions

$\phi \in \mathbb{R}^N \equiv \phi: Z \mapsto \mathbb{R}$; $Z = \{1, \dots, N\}$

ϕ: DOMAIN $\mapsto$ RANGE; $\phi \in$ Class (Domain, Range)

Today: Consider Domain $D \subset \mathbb{R}^d$ is bounded & open

Notation: $x = (x_1, \dots, x_d)^T \in \mathbb{R}^d$

(i) scalar field $f: D \mapsto \mathbb{R}$

(ii) vector field $\varphi: D \mapsto \mathbb{R}^d$

(iii) matrix field $A: D \mapsto \mathbb{R}^{d \times d}$

Ex: $\left.\begin{array}{l} f\varphi: D \mapsto \mathbb{R}^d \\ A\varphi: D \mapsto \mathbb{R}^d \end{array}\right\}$ vector fields

f, g scalar fields $\to fg: D \mapsto \mathbb{R}$

φ, ψ vector fields $\to \langle \varphi, \psi \rangle_{\mathbb{R}^d}: D \mapsto \mathbb{R}$

Calculus Notation:

- Given a scalar field $f: D \mapsto \mathbb{R}$, define the gradient $\nabla f: D \to \mathbb{R}^d$ as a vector field by
$\{\nabla f(x)\}_i = \frac{\partial f}{\partial x_i}(x) = f_{,i}(x)$
- Given a vector field $\varphi: D \mapsto \mathbb{R}^d$, define the divergence $\nabla.\varphi: D \mapsto \mathbb{R}$ as a scalar field by
$(\nabla.\varphi \mid x) = \sum_{i=1}^{d} \frac{\partial \varphi_i}{\partial x_i}(x) = \varphi_{i,i}$

Ex: Given $f: D \mapsto \mathbb{R}$ scalar field define $\Delta f: D \to \mathbb{R}$ scalar field by

$\Delta f = \nabla.(\nabla f)$ (= $\nabla^2 f$ in some texts)

$\Delta f(x) = f_{,ii} = \sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2}$

3.3 Flow in porous medium

$D \subset \mathbb{R}^d \quad d = \{1,2,3\}$

$A: D \mapsto \mathbb{R}^{d \times d}_{sym}$ permeability matrix (tensor)

$s: D \mapsto \mathbb{R}$ sources/sinks of fluid

$p: D \mapsto \mathbb{R}$ pressure in fluid

$v: D \mapsto \mathbb{R}^d$ velocity of fluid

Darcy Law $v = -A\nabla p$

Mass conservation $\nabla.v = s$

Combining

$$\text{⓪}\begin{cases} -\nabla.(A\nabla p) = s, & x \in D \\ p = 0, & x \in \partial D \end{cases}$$

... $\partial D = \bar{D} \setminus D$ - closure of set without set

↳ boundary of set D

Ex: $D = (0,1)$; $\bar{D} = [0,1]$; $\partial D = \{0,1\}$

Ex: $d = 1$

$-\frac{d}{dx}\left(a \frac{dp}{dx}\right) = s, \quad x \in (0,1)$

$p(0) = p(1) = 0$

Given A, s, find p

Definition

$\mathcal{L}q = -\nabla.(A\nabla q)$; $\mathcal{L}$: scalar field → scalar field

- (Need a function class; it will impose $q = 0$ on ∂D, and is at least C^2)
- $\mathcal{L}$ is linear: for any $\alpha, \beta \in \mathbb{R}$; $p, q \in$ function class

$\mathcal{L}(\alpha p + \beta q) = -\nabla.(A\nabla(\alpha p + \beta q))$
$= -\nabla.(\alpha A\nabla p + \beta \nabla q)$
$= -\alpha\nabla.(A\nabla p) - \beta\nabla.(A\nabla q)$
$= \alpha\mathcal{L}p + \beta\mathcal{L}q$

⓪ ≡ ①

① $\mathcal{L}p = s$; $p \in$ Function class

Analogous to: $Lp = s$ for $p, s \in \mathbb{R}^J$, $L \in \mathbb{R}^{J \times J}$ ②

which is equivalent to $\boxed{\langle Lp, v\rangle_{\mathbb{R}^J} = \langle s, v\rangle_{\mathbb{R}^J} \ \forall v \in \mathbb{R}^J}$ ②

$\int_D (\mathcal{L}p)(x)\, v(x)\,dx = \int_D s(x) v(x)\,dx \quad \forall v \in \mathcal{V}$

$\int_D -\nabla.(A\nabla p) v\,dx = \int_D s v\,dx \quad \forall v \in \mathcal{V}$

stoke's theorem → $\int_D \langle A\nabla p, \nabla v\rangle_{\mathbb{R}^d} dx = \int_D s v\,dx$

$:= B(p,v) \qquad := f(v)$

Alternate formulation: $\boxed{\text{Find } p \in \mathcal{V}: B(p,v) = f(v) \ \forall v \in \mathcal{V}}$ ③

③ ≢ ⓪ because now p must only be differentiable once

To solve ③ on a computer, we introduce $\mathcal{V}^J \subset \mathcal{V}$ defined by $\mathcal{V}^J = \text{span}\{\varphi_j\}_{j=1}^{J}$, each $\varphi_j \in \mathcal{V}$ user defined

$= \{v \in \mathcal{V} \mid \exists \alpha \in \mathbb{R}^J \ \& \ v(x) = \sum_{j=1}^{J} \alpha_j \varphi_j(x)\}$

$\boxed{\text{Find } p^J \in \mathcal{V}^J: B(p^J, v) = f(v) \quad \forall v \in \mathcal{V}^J}$ ③

$p^J(x) = \sum_{j=1}^{J} \alpha_j \varphi_j(x)$.

$\boxed{\text{Find } p^J: B(p^J, \varphi_j) = f(\varphi_j) \quad \forall j \in \{1,\dots,J\}}$ equivalent to ③

↳ can check basis rather than span

$B\left(\sum_{k=1}^{J} \alpha_k\varphi_k, \varphi_j\right) = f(\varphi_j) \quad j \in \{1,\dots,J\}$

$\sum_{k=1}^{J} B(\varphi_k, \varphi_j)\alpha_k = f(\varphi_j) \quad j \in \{1,\dots,J\}$

[$B(p,q) = B(q,p)$ b/c $A^T = A$ ... good to check ✱]

$\therefore \sum_{k=1}^{J} B(\varphi_j, \varphi_k)\alpha_k = f(\varphi_j) \quad j \in \{1,\dots,J\}$

$\equiv \boxed{L\alpha = r}$; $L_{jk} = B(\varphi_j, \varphi_k)$; $r_j = f(\varphi_j)$

↑ symmetric

Theorem $\|p - p^J\| \le \|p - q\| \quad \forall q \in \mathcal{V}^J$ as bounded error

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10/8 **L2: Control Theory**

① $x_{k+1} = Ax_k + Bu_k, \quad x_0 = a$

$x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$

$\{u_k\}_{k\in\mathbb{Z}^+}$, $\{x_k\}_{k\in\mathbb{Z}^+}$ $\quad \mathbb{Z}^+ = \{0,1,2\dots\} = \{0\} \cup \{\mathbb{N}\}$

$y_k = Dx_k$, $\{y_k\}_{k\in\mathbb{Z}^+}$ data

$y_k \in \mathbb{R}^b$, $D \in \mathbb{R}^{b\times n}$

Definition: The controllable set C is controllable if $C \equiv \mathbb{R}^n$ where
$C = \{a \in \mathbb{R}^n \mid \exists\, l \in \mathbb{Z}^+ \ \& \ \{u_j\}_{j=0}^{l-1} : x_l = 0\}$

Definition: $G = (B, AB, A^2B, \dots, A^{n-1}B) \in \mathbb{R}^{n\times nm}$

Theorem: Assume A^{-1} exists. Then the system is controllable iff rank $G = n$

Definition: The system is observable on $J' = \{0,\dots,J-1\}$ if, given $\{y_j\}_{j\in J'}$ and $\{u_j\}_{j\in J'}$ x_0 (and hence $\{x_{j+1}\}_{j\in J'}$) can be uniquely determined

Theorem: The system is observable iff $z_{k+1} = A^T z_k + D^T y_k$ is controllable

Q: Can we generalize ① to ∞ dimensional vector spaces?

Lemma: $x_k = A^k a + \sum_{j=0}^{k-1} A^{k-1-j} Bu_j$

Proof: $\boxed{k=0}$ $x_0 = a$ ✓

$\boxed{k \to k+1}$ Assume true for k. Then

$x_{k+1} = A\left(A^k a + \sum_{j=0}^{k-1} A^{k-1-j}Bu_j\right) + Bu_k$
$= A^{k+1}a + \sum A^{k-j} Bu_j + Bu_k$
$= A^{k+1}a + \sum_{j=0}^{k} A^{k-j}Bu_j$. Result at $(k+1)$

Result follows by induction.

Formulation of ① as a linear system:

$x_0 = a$
$x_1 - Ax_0 = Bu_0$
$x_2 - Ax_1 = Bu_1$
$\vdots$
$x_{k+1} - Ax_k = Bu_k$

$x = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_k \\ \vdots \end{pmatrix}$, $\mathbf{z} = \begin{pmatrix} a \\ u_0 \\ \vdots \\ u_k \\ \vdots \end{pmatrix}$

$A' = \begin{pmatrix} I & & & 0 \\ -A & I & & \\ & -A & I & \\ 0 & & & \ddots \end{pmatrix}$ $\quad B' = \begin{pmatrix} I & & & 0 \\ & B & & \\ & & B & \\ 0 & & & \ddots \end{pmatrix}$ $\quad I \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $A \in \mathbb{R}^{n\times n}$

① ⇔ $A'x = B'\mathbf{z}$

- Does this have a solution?
- Can we write $x = L\mathbf{z}$, $L = A'^{-1}B'$?
- Is L linear?

10/8 L2 cont'd:

Definition: $V = \{v_k\}_{k \in \mathbb{Z}^+}$, $v_k \in \mathbb{R}^\ell$, $\|\cdot\|$ on $\mathbb{R}^\ell$

Given $w = \{w_k\}_{k\in\mathbb{Z}^+}$, $w_k \in (0,\infty) = \mathbb{R}^+ \setminus \{0\}$ & $p \in [1,\infty)$

define $\ell^p_w(\mathbb{Z}^+;\mathbb{R}^\ell) = \{v : \|v\|_{\ell^p_w} < \infty\}$

$\|v\|^p_{\ell^p_w} = \sum_{j\in\mathbb{Z}^+} w_j \|v_j\|^p$ Note: $[\ell^{p_1}_w \supset \ell^{p_2}_w]$ for $p_1 > p_2$ ↳ only needed in ∞ dimensions

and define $\ell^\infty_w(\mathbb{Z}^+,\mathbb{R}^\ell) = \{v : \|v\|_{\ell^\infty_w} < \infty\}$ where

$\|v\|_{\ell^\infty_w} = \sup_{k\in\mathbb{Z}^+} w_k \|v_k\|$

Note: $\|\cdot\|$ on $\mathbb{R}^\ell$, the set $v : \|v\| < \infty$ does not depend on the definition of norm (for finite space).

Assumption 1: There are norms on vectors & matrices s.t.

① $\|Mv\| \le \|M\|\|v\|$ $\forall v \in \mathbb{R}^n$, $\forall M \in \mathbb{R}^{\ell\times n}$

② $\|MN\| \le \|M\|\|N\|$ $\forall M,N \in \mathbb{R}^{n\times n}$

Assumption 2: Given such a norm, $\exists \alpha \in (0,\infty) : \|A\| \le \alpha$

Theorem: Under assumptions 1 & 2, L is a bounded operator from $\underbrace{\mathbb{R}^n \times \ell^1(\mathbb{Z}^+;\mathbb{R}^m)}_{z=(a,u)}$ into $\underbrace{\ell^\infty(\mathbb{Z}^+;\mathbb{R}^n)}_{x}$ if $\alpha \in (0,1]$

[Notation]: $\ell^p(\mathbb{Z}^+;\mathbb{R}^\ell) \equiv \ell^p_w(\mathbb{Z}^+;\mathbb{R}^\ell)$, $w_j \equiv 1$

Proof (sketch) $x_k = A^k x_0 + \sum_{j=0}^{k-1} A^{k-1-j} B u_j$

$\|x_k\| \le \|A^k x_0\| + \sum_{j=0}^{k-1} \|A^{k-1-j} B u_j\|$ ~ Δ inequality on $\mathbb{R}^n$

$\|x_k\| \le \|A^k\|\|x_0\| + \sum \|A^{k-1-j}\|\|Bu_j\|$ from a^1

$\le \|A\|^k \|x_0\| + \sum \|A\|^{k-1-j} \|B\| \|u_j\|$ from a^1 & a^2

$\le \alpha^k \|a\| + \|B\| \sum \alpha^{k-1-j} \|u_j\|$ from a^2

$\le \|a\| + \|B\| \sum_{j=0}^{k-1} \|u_j\|$ by Thm. assumption

$\le \|a\| + \|B\| \sum_{j=0}^{\infty} \|u_j\| = \|a\| + \|B\| \|u\|_{\ell^1}$

↳ True for all k

$\therefore \sup_{k\in\mathbb{Z}^+} \|x_k\| \le \|a\| + \|B\|\|u\|_{\ell^1}$

$\|x\|_{\ell^\infty} \le \|a\| + \|B\| \|u\|_{\ell^1}$

Note: Function Spaces

Properties (Domain ; Range)

ex: $\ell^p(\mathbb{Z}^+;\mathbb{R}^\ell)$ discrete time

$C^?([0,\infty);\mathbb{R}^\ell)$ continuous time

Why/how?

TS: OPEN-NESS. $(x,\emptyset)$ in $\sqcup$, $\sqcup$ closed under arbitrary union, $\sqcup$ closed under finite intersection

MS: DISTANCE. $d(x,x) = 0$, $d(x,y) \ge 0$, $d(x,z) \le d(x,y) + d(y,z)$, $d(x,y) = d(y,x)$

~connected to openness §4.2~

MsS: SIZE/sign (of subsets): x in Σ, Σ closed under countable union, Σ closed under complements

PS: PROBABILITY

VS: VECTORS → Abstract

NVS: SIZE of elements of X

IPS: ANGLE between vectors

4.2 OPEN

Definition: 1) S is a subset of x, (x,d) a metric space, is open if $\forall x \in S$ $\exists \delta > 0$ s.t. if $y \in x$ and $d(x,y) < \delta$ then $y \in S$.

2) $B(a,r) = \{x \in X : d(a,x) < r\}$ Ball of radius $r \ge 0$ @ $x \in X$ ~ metric space

3) $B(a,r) = \{x \in X : \|a - x\| < r\}$ — " — (NVS)

4.3 Inter-relations:

Fact 1: Every metric space defines a topological space

Fact 2: Every PS is a measure space

Fact 3: Every NVS is a metric space : $d(a,b) = \|a-b\|$

Fact 4: Every IPS is a NVS : $\|a\|^2 = \langle a,a \rangle$

4.4 Examples: 1) Borel σ-algebra on (x,d) is smallest σ-algebra containing all open sets

2) $(\mathbb{R},\Sigma)$, Σ Borel-σ algebra ~ M_+ be collection of all prob meas. on $(\mathbb{R},\Sigma)$

3) $\mu(dx;m,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{|x-m|^2}{2\sigma^2}\right)dx$

For any $A \in \Sigma$: $\mu(A;m,\sigma) = \int_A \mu(dx;m,\sigma)$ Lebesgue

These $\mu(\cdot;m,\sigma)$ are probability measures : $\mathbb{P}^\mu(A) = \mu(A)$; $\mathbb{E}^\mu(f) = \int_{\mathbb{R}} f(x)\mu(dx)$ for any $f : \mathbb{R}\to\mathbb{R}$

↳ M_+ can be made a metric space as follows:

$d(\mu,\nu) = \sup_{f\in F} |\mathbb{E}^\mu f - \mathbb{E}^\nu f|$

$F = \{f : \mathbb{R} \to \mathbb{R} \mid \sup_{x\in\mathbb{R}} |f(x)| \le 1\}$

10/10 L3: Know your Spaces (x is a set) ~ $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ (scalar field)

- Topological Spaces (TS) $x, \sqcup$ $\sqcup$ collection of subsets of x ($\sqcup$ open sets)
- Metric Spaces (MS) x, d $d : X \times X \to \mathbb{R}$
- Measure Spaces (MsS) (x,Σ,μ) Σ collection of subsets of x, $\mu : \Sigma \to \mathbb{R} \cup \{\pm\infty\}$ (Σ is a σ-algebra)
- Probability Spaces (PS)   Measure space plus $\mu(x) = 1$, $\mu : \Sigma \to [0,1]$
- Vector Spaces (VS) $X, \mathbb{K}$, add in $X : X \times X \mapsto X$, multiply $\mathbb{K} \times X \mapsto X$, $0 \in X$, $1 \in \mathbb{K}$
- Normed VS (NVS)   VS + $\{\|\cdot\| : X \to \mathbb{R}\}$
- Inner product spaces (IPS)   VS + $\{\langle\cdot,\cdot\rangle : X \times X \to \mathbb{K}\}$

Q: Is M_+ a vector space?

$\mu, \nu \in M_+$

If M_+ were a vector space then $\mu+\nu \in M_+$. But $(\mu+\nu)(\mathbb{R}) = \mu(\mathbb{R}) + \nu(\mathbb{R}) = 1+1 = 2$

$\therefore \mu+\nu \notin M_+$

4.4: Normed Vector Spaces: X, $\|\cdot\|: X \to \mathbb{R}$

- $\|x\| \ge 0$, $\|x\|=0 \Leftrightarrow x=0$
- $\|\lambda x\| = |\lambda| \|x\|$ $\forall \lambda \in \mathbb{K}, x\in X$
- $\|x+y\| \le \|x\|+\|y\|$ $\forall x,y\in X$

Definition: Two norms $\|\cdot\|_a$ & $\|\cdot\|_b$ are equivalent if $\exists\ 0<c_1<c_2<\infty : \forall x \in X$

$c_1\|x\|_a \le \|x\|_b \le c_2 \|x\|_a$

$[\frac{1}{c_2}\|x\|_b \le \|x\|_a \le \frac{1}{c_1}\|x\|_b$

Theorem: All norms are equivalent in $\mathbb{R}^m$ Proof in notes

Ex: $v = \{v_0, v_1, v_2, \dots\}$ $v_k \in \mathbb{R}^m$

$\ell^p(\mathbb{Z}^+;\mathbb{R}^m) = \{v \mid \|v\|_{\ell^p} < \infty\}$

$$\|v\|_{\ell^p}^p = \sum_{k=0}^\infty \|v_k\|_{\mathbb{R}^m}^p$$

Fact: $\ell^p(\mathbb{Z}^+;\mathbb{R}^m)$ is the same set of sequences, independently of choice of $\|\cdot\|_{\mathbb{R}^m}$

Let $\|\cdot\|_a, \|\cdot\|_b$ be any two norms on $\mathbb{R}^m$. We show that $\ell^p(\mathbb{Z}^+;\mathbb{R}^m)$ is the same

$c_1^p\|v_k\|_a^p \le \|v_k\|_b^p \le c_2^p\|v_k\|_a^p$

key: c_1, c_2 are independent of k

$$c_1^p \sum_{k=0}^n \|v_k\|_a^p \le \sum_{k=0}^n \|v_k\|_b^p \le c_2^p \sum_{k=0}^n\|v_k\|_a^p$$

Now let $n\to\infty$

$$\boxed{c_1^p \sum_{k=0}^\infty \|v_k\|_a^p \le \sum_{k=0}^\infty \|v_k\|_b^p \le c_2^p \sum_{k=0}^\infty\|v_k\|_a^p}$$

Being finite in one norm implies being finite in any other norm.

10/15 L4: Chapter 5: Banach & Hilbert Spaces

$(V, \|\cdot\|)$ is a normed vector space

Definition: $v \in V$ is a limit point of sequence $\{v_n\}_{n\in\mathbb{N}}$ if $\|v_n - v\| \to 0$ as $n\to\infty$. We say v_n converges to v

Definition: $S \subset V$ is closed if $\{v_n\}_{n\in\mathbb{N}}$ is a sequence in S (i.e. $v_n \in S, n\in\mathbb{N}$) & v_n converges to v implies $v\in S$

Notation: $\bar{S} = S + \{$all limit points in $V\setminus S\}$

Example: $B_V(w,r) = \{u \in V \mid \|u-w\| < r\}$

(open ball in V at $w\in V$ of radius $r\in\mathbb{R}^+$)

$v_n = w + (1-\frac{1}{n})z$ $\|z\| = r$; $w, z\in V$

$v = w+z$; $\|v_n - v\| = \|\frac{1}{n}z\| = \frac{1}{n}r$ $\therefore \|v_n - v\|\to 0$ as $n\to\infty$

$\therefore v$ is the limit point of $\{v_n\}$

$B_V(w,r)$ is not closed b/c $\|v-w\| = \|z\| = r$ $\therefore v \notin B_V(w,r)$

However, $\|v_n - w\| = (1-\frac1n)\|z\| = (1-\frac1n) r < r$ $\therefore v_n \in B_V(w,r)$

~The only possible limit points of sequences $\{v_n\}$ must be on boundary

Definition: $\{v_n\}_{n\in\mathbb{N}}$ in $(V,\|\cdot\|)$ is Cauchy if $\forall \varepsilon>0\ \exists N=N(\varepsilon): \forall n,m\ge N$, $\|v_n - v_m\| < \varepsilon$

Definition: $(V,\|\cdot\|)$ is complete if every Cauchy sequence is convergent to a limit $v\in V$

Definition: Banach Space ≡ complete NVS

Hilbert Space ≡ complete IPS

Important Example: $\mathbb{R}^n$ with any norm is complete. $(\mathbb{R}^n, \|\cdot\|)$ is a Banach space. Similarly $\mathbb{R}^n$ with Euclidean inner-product is a Hilbert space

Example: $D \subset \mathbb{R}^d$ bounded, open. Σ Borel σ-algebra; (D,Σ,μ) is the probability space (e.g. $d=2$, μ is Lebesgue measure, probability = normalized area

$L^p_\mu(D;\mathbb{R}^m) = \{f : \|f\|_{L^p_\mu} < \infty\}$ ⇝ Banach Space

$$\|f\|^p_{L^p_\mu} = \int_D \|f(x)\|^p_{\mathbb{R}^m} \underbrace{\mu(dx)}_{dx \text{ for Riemann integration}}$$

Hilbert if $p=2$:

$$\langle a,b\rangle_{L^2_\mu} = \int_D \langle a(x), b(x)\rangle \mu(dx)$$

$\ell^p(\mathbb{N};\mathbb{R})$ is a Banach space

$\|f\|^p_{\ell^p} = \sum_{j=1}^\infty |f_j|^p$, $\ell^p(\mathbb{N};\mathbb{R}) := \{f:\mathbb{N}\to\mathbb{R} \mid \|f\|_{\ell^p}<\infty\}$

Definition: $p,q\in[1,\infty] := [1,\infty)\cup\{\infty\}$ are conjugate if $\frac1p+\frac1q = 1$

Lemma: $\frac1p + \frac1q = 1 \Leftrightarrow p+q = pq \Leftrightarrow p = (p-1)q$

Lemma: (Minkowski) If $1\le p\le\infty$ & $v,w\in\ell^p$ then

$\|v+w\|_{\ell^p} \le \|v\|_{\ell^p} + \|w\|_{\ell^p}$

Theorem: ℓ^p is a Banach space

Proof: • $\|\cdot\|_{\ell^p}$: real sequences into $\mathbb{R}^+$.

- $\|f\|_{\ell^p} = 0 \Leftrightarrow \sum_{j=1}^\infty |f_j|^p = 0 \Leftrightarrow f_j = 0\ \forall j\in\mathbb{N} \Leftrightarrow f=0$
- $\alpha\in\mathbb{R}$ $\|\alpha f\|^p_{\ell^p} = \sum_{j=1}^\infty|\alpha f_j|^p = \sum_{j=1}^\infty |\alpha|^p|f_j|^p = |\alpha|^p\sum_{j=1}^\infty|f_j|^p$ $= |\alpha|^p\|f\|^p_{\ell^p}$
- Minkowski → Triangle inequality

We have shown that $\{f:\mathbb{N}\to\mathbb{R}, \|\cdot\|_{\ell^p}\}$ Normed Vector Space

- Remains to show that every cauchy sequence has a limit in ℓ^p ⇝ completeness

Idea: Use finite dimensional sequences (where we have completeness)

$\{v^{(n)}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in ℓ^p if, for any $\varepsilon>0$ $\exists N=N(\varepsilon): \forall n,m\ge N$ $\|v^{(n)} - v^{(m)}\|_{\ell^p} < \varepsilon$

[for fixed n, $v^{(n)}$ is itself a sequence ≡ A function $\mathbb{N}\to\mathbb{R}$]

⊛ $\boxed{\sum_{j=1}^\infty |v_j^{(n)} - v_j^{(m)}|^p < \varepsilon^p}$ ↳ $v_j^{(n)}$ is a $\mathbb{R}$ #.

We want to identify $v'\in\ell^p$: $\|v^{(n)} - v'\|_{\ell^p}\to 0$ as $n\to\infty$

⊛ ⇒ for any $j\in\mathbb{N}$: $|v_j^{(n)} - v_j^{(m)}| < \varepsilon$ $\forall n,m\ge N(\varepsilon)$

⇒ $v_j^{(n)} \to v_j' \in\mathbb{R}$

Suggest that $v' := (v_1', v_2', v_3', \dots) \in$ space of sequences

We try & prove that $v^{(n)} \xrightarrow{\ell^p} v'$ as $n\to\infty$

Need to show: a) $v'\in\ell_p$ & b) $\|v^{(n)}-v'\|_{\ell^p}\to0$ as $n\to\infty$

By ⊛ $\forall n,m\ge N$ $\sum_{j=1}^M |v_j^{(n)}-v_j^{(m)}|^p \le \sum_{j=1}^\infty |v_j^{(n)} - v_j^{(m)}|^p < \varepsilon^p$ for any $M\in\mathbb{N}$

$v_j^{(m)} \to v_j'$ as $m\to\infty$

$\forall n\ge N$ $\sum_{j=1}^M |v_j^{(n)} - v_j'|^p \le \varepsilon^p < (2\varepsilon)^p$

$\forall n\ge N$ $\left(\sum_{j=1}^\infty |v_j^{(n)} - v_j'|^p\right)^{1/p} < 2\varepsilon$ ⇝ $\forall n\ge N(\varepsilon)$ $\left(\sum_{j=1}^\infty |v_j^{(n)} - v_j'|^p\right)^{1/p} < 3\varepsilon$

i.e. for any $\varepsilon>0$ $\exists N(\varepsilon): \forall n\ge N(\varepsilon)$ $\|v^{(n)} - v'\|_{\ell^p} < 3\varepsilon$ ⊙

By Minkowski: $\|v'\|_{\ell^p} = \|v^{(n)} + v' - v^{(n)}\|_{\ell^p} \le \underbrace{\|v^{(n)}\|_{\ell^p}}_{<\infty} + \underbrace{\|v'-v^{(n)}\|_{\ell^p}}_{<3\varepsilon}$

i.e. $v'\in\ell_p$.

⊙ shows $v^{(n)}\to v'$ as $n\to\infty$

$\therefore$ ℓ^p is complete $\therefore$ BANACH

10/17 **L6:**

C^k spaces: $\alpha \in (\mathbb{Z}^+)^n$ multi-index

$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n = \|\alpha\|_1$ ⋮ $\partial_\alpha = (\partial_1, \dots, \partial_n)$, $\partial_j = \frac{\partial}{\partial x_j}$

$x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$ ⋮ $\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}$

Examples $u: \mathbb{R}^n \to \mathbb{R}$

$$\sum_{|\alpha|=1} \partial^\alpha u = \sum_{j=1}^n \partial_j u = \nabla \cdot u$$

$n=2)$ $$\sum_{|\alpha|=2} \partial^\alpha u = \frac{\partial^2}{\partial x_1^2}u + \frac{\partial^2}{\partial x_2^2}u + 2\frac{\partial^2}{\partial x_1 \partial x_2}u$$

Let $D \subseteq \mathbb{R}^d$ be an open subset ($D = \mathbb{R}^d$ included)

Definition: $C(\bar{D};\mathbb{R}) = \{u: D \to \mathbb{R}$, continuous on $\bar{D}$ & $\|u\|_C < \infty\}$

$\|u\|_C = \sup_{x \in D} |u(x)|$ Banach space

Definition: $C^k(\bar{D};\mathbb{R}) = \{u: D \to \mathbb{R} \mid \partial^\alpha u \in C(\bar{D};\mathbb{R})\ \forall |\alpha| \le k, \|u\|_{C^k} < \infty\}$

$\|u\|_{C^k} = \sup_{x\in D} \sum_{0 \le |\alpha| \le k} |\partial^\alpha u(x)|$ Banach space

Definition: $C^\infty(\bar{D};\mathbb{R}) = \bigcap_{k=0}^\infty C^k(\bar{D};\mathbb{R})$

Can change functions on sets of measure zero and can choose best case

$L^\infty(D;\mathbb{R}) = \{u: D \to \mathbb{R}, \|u\|_{L^\infty} < \infty\}$; $\|u\|_{L^\infty} = \operatorname{ess\,sup}_{x \in D} |u(x)|$ Banach Space

Ex: $u(x) = 1$, $x \in [0,1]$ is in $C([0,1];\mathbb{R})$ and in $L^\infty((0,1);\mathbb{R})$

$u(x) = \begin{cases} 0 & x \in [0, \frac{1}{2}) \\ 1 & x \in [\frac{1}{2}, 1] \end{cases}$ is not in $C([0,1];\mathbb{R})$ but is in $L^\infty((0,1);\mathbb{R})$

$u(x) = \begin{cases} \frac{1}{x} & x \in (0,1] \\ \infty & x = 0 \end{cases}$ is not in $C([0,1];\mathbb{R})$ or in $L^\infty((0,1);\mathbb{R})$

$u(x) = \begin{cases} 0 & \text{in } x \in (0,1] \setminus \{\frac{1}{2}\} \\ \infty & \text{at } x = \frac{1}{2} \end{cases}$ is not in $C([0,1];\mathbb{R})$ but is in $L^\infty((0,1);\mathbb{R})$

↳ can change ∞ to anything at $x = \frac{1}{2}$

Weak Derivative (READ): $1 \le p \le \infty$, $k \in \mathbb{N}$

$W^{k,p}(D;\mathbb{R}) = \{u: D \to \mathbb{R} \mid \partial^\alpha u \in L^p(D;\mathbb{R}^{|\alpha|}), 0 \le |\alpha| \le k\}$ Banach Space

$$\|u\|^p_{W^{k,p}} = \sum_{0 \le |\alpha| \le k} \|\partial^\alpha u\|^p_{L^p}$$

$H^k(D;\mathbb{R}) = W^{k,2}(D;\mathbb{R})$ is a Hilbert space

$$\langle u, v\rangle_{H^k} = \sum_{0 \le |\alpha| \le k} \langle \partial^\alpha u, \partial^\alpha v\rangle_{L^2(D;\mathbb{R}^{|\alpha|})}$$

Examples from L5: $\ell^p = \ell^p(\mathbb{N};\mathbb{R})$ $v_j = j^{-s}$, $s \ge 0$

$s = 0$ $v \in \ell^\infty$, $\|v\|_{\ell^\infty} = 1$; $v \notin \ell^p$, $p \in [1, \infty)$

$s > 0$ $\|v\|^p_{\ell^p} = \sum_{j=1}^\infty j^{-sp} < \infty$ iff $sp > 1$; $v \in \ell^p$ $p \in (\frac{1}{s}, \infty]$

$v \notin \ell^p$ $p \in [1, \frac{1}{s}]$

Ex: $D = B_0(1)$ in $\mathbb{R}^d$

$B_0(1) = \{x \in \mathbb{R}^d, |x| < 1\}$ where $|\cdot|$ = Euclidean norm

$v(x) = |x|^{-s}$, $s \ge 0$

$s = 0$ $v \in L^p(D;\mathbb{R})$ $\forall p \in [1, \infty]$

$d > s > 0$ $\|v\|^p_{L^p} = \int_D |x|^{-sp}\,dx = c\int_0^1 |r|^{-sp} r^{d-1}\,dr = c\int_0^1 \frac{1}{|r|^{1+sp-d}}\,dr$

$< \infty$ iff $1 + sp - d < 1 \to p < d/s$

$\therefore$ $v \in L^p(D;\mathbb{R})$ $\forall p \in [1, d/s)$; $v \notin L^p(D;\mathbb{R})$ $\forall p \in [d/s, \infty]$

$s \ge d$ $v \notin L^p(D;\mathbb{R})$ $\forall p \in [1, \infty]$

Fourier Series: $D = (0,1)$; $u: D \to \mathbb{R}$ $L^2 = L^2(D;\mathbb{R})$

$\varphi_j(x) = \sqrt{2}\sin(j\pi x)$, $j \in \mathbb{N}$ $\langle a, b\rangle_{L^2} = \int_0^1 a(x)b(x)\,dx$

$\langle \varphi_j, \varphi_k\rangle = \delta_{jk} = \begin{cases} 1 & j = k \\ 0 & j \ne k \end{cases}$

~ Spectral Theorem (ch. 13) ~

$\boxed{u(x) = \sum_{j=1}^\infty a_j \varphi_j(x)}$ ⊖; $\langle u, \varphi_\ell\rangle = \sum_{j=1}^\infty a_j\langle \varphi_j, \varphi_\ell\rangle = \sum_{j=1}^\infty a_j \delta_{j\ell} = a_\ell$

$\boxed{a_\ell = \langle \varphi_\ell, u\rangle, \ell \in \mathbb{N}}$ ⊛

$$\|u\|^2_{L^2} = \sum_{j=1}^\infty \sum_{k=1}^\infty a_j a_k \underbrace{\langle \varphi_j, \varphi_k\rangle}_{\delta_{jk}} = \sum_{j=1}^\infty a_j^2 = \|a\|^2_{\ell^2}$$

- ⊛ Can be viewed as a linear transformation $F: L^2 \mapsto \ell^2$
- ⊖ Can be viewed as a linear transformation $F^{-1}: \ell^2 \mapsto L^2$

check linearity

- $F^{-1}F = I$ on L^2

$\frac{du}{dx} = u'(x) = \sum_{j=1}^\infty a_j j\pi\sqrt{2}\cos(j\pi x)$; $\|u'(x)\|^2_{L^2} = \sum_{j=1}^\infty a_j^2 j^2\pi^2$

$\|u\|^2_{L^2} + \|u'\|^2_{L^2} = \sum_{j=1}^\infty (1 + j^2\pi^2)a_j^2 = \|a\|^2_{\ell^2_w}$; $w = (w_1, w_2, \dots)$ $w_j = (1 + j^2\pi^2)$

$$\boxed{\|u\|^2_{H_0^1} = \|a\|^2_{\ell^2_w}}$$

10/22 **L7:** Linear Operators

Example 1: · Fourier Series ($F: L^2 \to \ell^2$; $F^{-1}: \ell^2 \to L^2$)

· Fourier Transform ($F: L^2 \to L^2$; $F^{-1}: L^2 \to L^2$)

Matrices in $\mathbb{R}^{n\times n}$:

$A \in \mathbb{R}^{n\times n}$ $A \in \mathcal{L}(\mathbb{R}^n;\mathbb{R}^n)$

$A, B \in \mathbb{R}^{n\times n}$ $\alpha, \beta \in \mathbb{R}$ $\to \alpha A + \beta B \in \mathbb{R}^{n\times n}$

The set $X = \{$all $A \in \mathbb{R}^{n\times n}\}$ forms a vector space

$\left(\begin{array}{l} + \text{ w/ } A,B \\ \times \text{ w/ } \alpha, A \end{array}\right)$ vector space $\Big\{$ (o w/ A, B) algebra

Norms on $\mathbb{R}^{n\times n}$:

Ex: $\langle A, B\rangle = \sum_{i,j=1}^n a_{ij}b_{ij}$

- $\|A\|_F = \left(\sum_{i,j=1}^n |a_{ij}|^2\right)^{1/2}$
- $\|A\|_{max} = \max_{1 \le i,j \le n} |a_{ij}|$
- Any norm in $\mathbb{R}^{n^2}$ can be used as a $\|\cdot\|$ on a matrix

Induced norms (operator norms)

- $\|A\|_\infty = \max_{1 \le \ell \le n} \sum_{m=1}^n |a_{\ell m}|$
- $\|A\|_1 = \max_{1 \le m \le n} \sum_{\ell=1}^n |a_{\ell m}|$
- $\|A\|_2 = \rho(A^T A)$ spectral radius

Operator Norm:

Let $A \in \mathbb{R}^{n\times n}$, $\|\cdot\|_a$, $\|\cdot\|_b$ be norms on $\mathbb{R}^n$. Then the norm on $\mathbb{R}^{n\times n}$ induced by these norms on $\mathbb{R}^n$ is

$$\|A\| = \sup_{u \ne 0} \frac{\|Au\|_b}{\|u\|_a} = \sup_{\|u\|_a = 1} \|Au\|_b$$

Ex: $\|A\|_\infty$ is induced by $\|\cdot\|_a = \|\cdot\|_b = \|\cdot\|_\infty$

$\|A\|_1$ is induced by $\|\cdot\|_a = \|\cdot\|_b = \|\cdot\|_1$

Ex: $\|A\|_2 = \sup \frac{\|Au\|_2}{\|u\|_2} \rightsquigarrow \|A\|_2^2 = \sup_{\|u\|_2^2 = 1} \|Au\|_2^2 = \sup_{\|u\|_2^2=1} \langle Au, Au\rangle$

$= \sup_{\langle u,u\rangle = 1} \langle u, A^T A u\rangle$ $\rightsquigarrow A^T A \varphi_j = \lambda_j^2 \varphi_j$ Assuming distinct λ for simplicity

$\{\varphi_j\}_{j=1}^n$ form orthonormal basis for $\mathbb{R}^n$ w.r.t. $\langle\cdot,\cdot\rangle$

$u = \sum_{j=1}^n u_j \varphi_j$; $u_j = \langle u, \varphi_j\rangle$; $\langle u, u\rangle = \sum_{j=1}^n u_j^2$; $\langle u, A^T A u\rangle = \sum_{j=1}^n \lambda_j^2 u_j$

(cont'd on back)

10/22 L7: cont'd

$\|A\|_2^2 = \sup_{\sum_{j=1}^n u_j^2 = 1} \sum_{j=1}^n u_j^2 \lambda_j^2$

LAGRANGE MULTIPLIERS

$J(\{u_j\}_{j=1}^n, \gamma) = \sum_{j=1}^n u_j^2 \lambda_j^2 - \gamma\left(\sum_{j=1}^n u_j^2 - 1\right)$

$\left.\begin{array}{l}\frac{\partial J}{\partial u} = 2u_j(\lambda_j^2 - \gamma) = 0 \rightarrow \gamma = \lambda_j^2 \text{ or } u_j = 0 \\ \frac{\partial J}{\partial \gamma} = \sum_{j=1}^n u_j^2 = 1\end{array}\right\} *$

$\gamma = \lambda_n^2, u_j = 0 \; j = 1, \dots, n-1 ; u_n = 1$ is a solution of $*$

Bounded Linear Operators:

$(V, \|\cdot\|_V), (W, \|\cdot\|_W)$ NVS over $\mathbb{R}$

Definition: $L: V \to W$ is linear if

$L(\alpha v_1 + \beta v_2) = \alpha L v_1 + \beta L v_2 \quad \forall \alpha, \beta \in \mathbb{R}, v_1, v_2 \in V, Lv_1, Lv_2 \in W$

Definition: $L: V \to W$ is bounded if

$\exists k > 0 : \forall u \in V \; \|Lu\|_W \le k\|u\|_V$ ⓪

Definition: $\mathcal{L}(V, W) = \{$All bounded linear operators from L to W$\}$

Ex: Let $V = W = \mathbb{R}^n$. $\mathcal{L}(V, W) = \{$All $A \in \mathbb{R}^{n \times n}$ matrices$\}$

$L \in \mathcal{L}(V, W)$

$\|L\| = \sup_{u \ne 0} \frac{\|Lu\|_W}{\|u\|_V}$ ① $\therefore$ for any $u \in V$ $\frac{\|Lu\|_W}{\|u\|_V} \le \|L\|$

$\Rightarrow \|Lu\|_W \le \|L\| \|u\|_V \quad \forall u \in V$

Lemma: Definition ① of $\|L\|$ is equivalent to defining

$\|L\| = \{\inf k \in \mathbb{R}^+ | ⓪ \text{ holds}\}$

$\boxed{\text{* This development is only true for operator (induced) norms!!}}$

Proposition: $\mathcal{L}(V, W)$ is a normed vector space when equipped w/ ①

Proof: (Sketch) $L_1, L_2 \in \mathcal{L}(V, W)$ $a, b \in \mathbb{R}$. Then $\alpha, \beta \in \mathbb{R}$, $v_1, v_2 \in V$

$(aL_1 + bL_2)(\alpha v_1 + \beta v_2) = aL_1(\alpha v_1 + \beta v_2) + bL_2(\alpha v_1 + \beta v_2)$

... Look at notes to complete

- $\|L\| = 0 \Leftrightarrow \|Lv\|_W = 0 \quad \forall v \ne 0$
  $\Leftrightarrow Lv = 0 \quad \forall v \ne 0$
  $\Leftrightarrow L \equiv 0$
- $\|\alpha L\| = \sup_{\|u\|_V = 1} \|\alpha L v\|_W = \sup_{\|u\|_V = 1} |\alpha| \|Lv\|_W = |\alpha| \sup_{\|u\|_V = 1} \|Lv\|_W = |\alpha| \|L\|$
- $\|T + S\| = \sup_{\|u\|_V = 1} \|(T+S)v\|_W = \sup_{\|u\|_V = 1} \|Tv + Sv\|_W$
  $\le \sup_{\|u\|_V = 1} \{\|Tv\|_W + \|Sv\|_W\} \le \sup_{\|u\|_V = 1} \|Tv\|_W + \sup_{\|u\|_V = 1} \|Sv\|_W = \|T\| + \|S\|$

Theorem: $(V, \|\cdot\|_V)$ is a NVS
$(W, \|\cdot\|_W)$ is a Banach space
Then $\mathcal{L}(V, W)$ is a Banach space

~

$L \in \mathcal{L}(V, W)$; $\|L\| = \sup \frac{\|Lu\|_W}{\|u\|_V}$ $\left[\|Lu\|_W \le \|L\| \|u\|_V\right]$ $\|AB\| \le \|A\|\|B\|$

Banach Algebra: $V = W$ Banach, $\|\cdot\|$, $\mathcal{L}(V, V)$

$A, B \in \mathcal{L}(V, V)$ define $(A \circ B)u := A(Bu) \quad \forall u \in V$

Definition: (Banach Algebra)
- $A \circ (B \circ C) = (A \circ B) \circ C$
- $\|AB\| \le \|A\| \|B\|$
- $\exists E \in \mathcal{L}(V, V)$ $A \circ E = E \circ A = A$

Ex: $V = \mathbb{R}^n$, $\|\cdot\|$ induced norm $E = I \in \mathbb{R}^{n \times n}$

Exercise: $\|AB\| = \sup_{\|u\| = 1} \|ABu\| \le \sup_{\|u\| = 1} \|A\| \|Bu\| = \|A\| \cdot \|B\|$

$\|A^k\| \le \|A\|^k$ $\triangle$

10/24 L8: Ch 8+12 Duality & Riesz Representation Theorem

Definition: Two Banach spaces $(X^*, \|\cdot\|_{X^*})$ $(Y, \|\cdot\|_Y)$ are isometrically isomorphic ($\cong$) if $\exists$ an invertible linear transformation between X^* and Y which preserves the norm.

Roughly: $B: X \to Y$; $B^{-1}: Y \to X \rightsquigarrow \|Bx\|_Y = \|x\|_X$, $\|B^{-1}y\|_X = \|y\|_Y$

Why use this? ; $(X, \|\cdot\|_X)$ Banach, $(\mathbb{R}, \|\cdot\|)$ Banach

$X^* = \mathcal{L}(X, \mathbb{R})$, $\|\ell\|_{X^*} = \sup_{x \ne 0} \frac{|\ell(x)|}{\|x\|_X} \rightsquigarrow X^*$ is a Banach space

(Linear map onto the reals := Dual space of X)

Goal: $X^* \cong Y$

Ex: Fourier Series: $I = (0, 1)$

$L^2 = (L^2(I; \mathbb{R}), \langle \cdot, \cdot \rangle_{L^2}, \|\cdot\|_{L^2})$, $\langle a, b \rangle_{L^2} = \int_I a(x) b(x) dx$

$\ell^2 = (\ell^2(\mathbb{N}; \mathbb{R}), \langle \cdot, \cdot \rangle_{\ell^2}, \|\cdot\|_{\ell^2})$, $\langle \alpha, \beta \rangle_{\ell^2} = \sum_{j \in \mathbb{N}} \alpha_j \beta_j$

$\varphi_j(x) = \sqrt{2} \sin(j \pi x)$, $\langle \varphi_j, \varphi_k \rangle_{L^2} = \delta_{jk}$

For $u \in L^2$ define $u_j = \langle u, \varphi_j \rangle$, $j \in \mathbb{N}$ ⓐ

For $v \in \ell^2$ define $v = \sum_{j \in \mathbb{N}} v_j \varphi_j(x)$ ⓑ

$\left.\begin{array}{l}ⓐ \text{ defines } B: L^2 \to \ell^2 \\ ⓑ \text{ defines } C: \ell^2 \to L^2\end{array}\right\}$ inverses of each other

$\|Bu\|_{\ell^2} = \|u\|_{L^2}$; $\|B^{-1}v\|_{L^2} = \|v\|_{\ell^2}$ (Parseval)

Behind this: if $u = \sum_{j=1}^\infty u_j \varphi_j$, $u_j = \langle u, \varphi_j \rangle$

$\|u\|_{L^2}^2 = \langle \sum_{j=1}^\infty u_j \varphi_j, \sum_{k=1}^\infty u_k \varphi_k \rangle = \sum_{j,k=1}^\infty \langle \varphi_j, \varphi_k \rangle u_j u_k = \sum_{j=1}^\infty u_j^2 = \|Bu\|_{\ell^2}^2$

Then we say that B is an isometric isomorphism b/w L^2 & ℓ^2

Example: $X = \mathbb{R}^n$, $\|u\|_p^p = \sum_{j=1}^n u_j^p \to p = 2$, $\langle a, b \rangle = \sum_{j=1}^n a_j b_j$

p=2 $f: \mathbb{R}^n \to \mathbb{R}$ linear map ; $f \in X^*$

$e^{(j)} = (0, \dots, 0, 1, 0, \dots, 0)^T \in \mathbb{R}^n$
↑ j-th entry

Any $x \in \mathbb{R}^n$ can be written as $x = \sum_{j=1}^n x_j e^{(j)}$, $x = (x_1, \dots, x_n)^T$

Thus $f(x) = \sum_{j=1}^n x_j f(e^{(j)})$

Define $y_j = f(e^{(j)})$, $y = (y_1, \dots, y_n)^T \rightsquigarrow f(x) = \langle x, y \rangle$

Any linear functional has associated with it a $y \in \mathbb{R}^n$

$f \in X^* \longmapsto y \in \mathbb{R}^n$ $\{ y \in \mathbb{R}^n \longmapsto f \in X^* := f(x) = \langle x, y \rangle$

$|f(x)| \le \|x\|_2 \|y\|_2$ (by Cauchy-Schwarz)

$\|f\|_{X^*} = \sup_{x \ne 0} \frac{|f(x)|}{\|x\|_2} \le \|y\|_2$ ⓒ

$|f(y)| = \langle y, y \rangle = \|y\|_2^2$ ⓒ & ⓓ imply $\|f\|_{X^*} = \|y\|_2$

$\frac{|f(y)|}{\|y\|_2} = \|y\|_2$ ⓓ $\Longrightarrow X^* \cong X = (\mathbb{R}^n, \|\cdot\|_2)$

Example: $I = (0, 1)$; $X = L^2(I; \mathbb{R})$; $y \in X$

Define $f: X \to \mathbb{R}$ by $f(x) = \langle x, y \rangle_{L^2} = \int_0^1 x(t) y(t) dt$ (linear)

$|f(x)| = \left|\int_0^1 x(t) y(t) dt\right| = |\langle x, y \rangle_{L^2}| \le \|x\|_{L^2} \|y\|_{L^2}$

$\sup_{x \ne 0} \frac{|f(x)|}{\|x\|_{L^2}} \le \|y\|_{L^2}$, $|f(y)| = \|y\|_{L^2}^2 \Longrightarrow \|f\|_{X^*} = \|y\|_{L^2}$

Conjecture $X^* \cong X$. Missing: need to show that any $f \in X^* = \mathcal{L}(X; \mathbb{R})$ can be written as: $f(x) = \langle x, y \rangle$ for some $y \in L^2$.

Lemma: If H is a Hilbert space (over $\mathbb{R}$) and $y \in H$ then $f_y: H \to \mathbb{R}$ defined by $f_y(x) = \langle x, y \rangle$ is an element of H^* & $\|f_y\|_{H^*} = \|y\|_H$

Riesz Representation theorem: $H^* \cong H$

Proof: (Sketch) Content is showing that for any $f \in \mathcal{L}(H; \mathbb{R})$

$\exists y \in H : f(x) = \langle x, y \rangle$

CMS 107

10/24 L8 cont'd:

What is special about Hilbert space?

- $X=\mathbb{R}^n$ $x,y\in\mathbb{R}^n$

  $|\sum_{j=1}^n x_jy_j|\le\|x\|_2\|y\|_2$ (Cauchy-Schwartz)

  $|\sum x_jy_j|\le\|x\|_p\|y\|_q$ (Hölder) $p^{-1}+q^{-1}=1$ $p\in[1,\infty]$

  $(\mathbb{R}^n,\|\cdot\|_p)^*\cong(\mathbb{R}^n,\|\cdot\|_q)$ $p\in[1,\infty]$

- $X=\mathbb{R}^\infty$, $\|x\|_{\ell^p}^p=\sum_{j=1}^\infty|x_j|^p$ $p\in[1,\infty)$; $\|x\|_{\ell^\infty}=\sup_{j\in\mathbb{N}}|x_j|$

  $\to(\mathbb{R}^\infty,\|\cdot\|_{\ell^p})^*\cong(\mathbb{R}^\infty,\|\cdot\|_{\ell^q})$ $p\in[1,\infty)$ $\to(\ell_p)^*\cong\ell^q$

  $(\mathbb{R}^\infty,\|\cdot\|_{\ell^\infty})^*\supset(\mathbb{R}^\infty,\|\cdot\|_{\ell^1})$ $\to\ell^\infty\supsetneq\ell^1$

  ↑ strict

- $c_0=\{u\in\ell^\infty:u_j\to0\text{ as }j\to\infty\}\to c_0\subset\ell^\infty$ (strict)

  $X:=c_0$; $\|\cdot\|_{c_0}=\|\cdot\|_{\ell^\infty}$

  $$\underbrace{(\mathbb{R}^\infty,\|\cdot\|_{c_0})^*}_{c_0}\cong\underbrace{(\mathbb{R}^\infty,\|\cdot\|_{\ell^1})}_{\ell^1}$$

10/25 Recitation: Fourier Transform

Definition: Let $u\in L^1(\mathbb{R}^d;\mathbb{C})$. Then the Fourier transform of u is

$$\hat u(y)=\frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}e^{-ix\cdot y}u(x)\,dx$$

and the inverse transform is:

$$\check u(y)=\frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}e^{ix\cdot y}u(x)\,dx$$

Notice: $|\hat u(y)|\le\frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}|u(x)|dx=c\|u\|_{L^1}<\infty$

Furthermore, $\|\hat u\|_{L^\infty}=\operatorname{ess\,sup}_{y\in\mathbb{R}^d}|\hat u|\le c\|u\|_{L^1}<\infty$ so $\hat u\in L^\infty(\mathbb{R}^d;\mathbb{C})$

Definition: Let $u\in L^p(\mathbb{R}^d)$ and $v\in L^q(\mathbb{R}^d)$ for $1\le p,q\le\infty$. Then define the convolution between u and v by

$$(u*v)(y)=\int_{\mathbb{R}^d}u(x)v(y-x)dx=\int_{\mathbb{R}^d}u(y-x)v(x)dx$$

Lemma ①: The following holds:

(i) If p,q are s.t. $R=(\frac1p+\frac1q-1)^{-1}\ge1$ then $(u*v)\in L^R(\mathbb{R}^d)$.

(ii) If p,q are Hölder conjugates, then $(u*v)\in C(\mathbb{R}^d)$

Lemma ②: Let $u\in C(\mathbb{R}^d)$ then

$$u(y)=\lim_{\varepsilon\to0^+}\frac{1}{(4\pi)^{d/2}}\int_{\mathbb{R}^d}u(x)e^{\frac{-|x-y|^2}{4\varepsilon}}$$

Theorem: [Plancharel] Suppose $u\in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$ then $\hat u,\check u\in L^2(\mathbb{R}^d)$ and $\|u\|_{L^2}=\|\hat u\|_{L^2}=\|\check u\|_{L^2}$. Consequently, we can extend $\hat{\ },\check{\ }:L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)$

Proof: Define $v(x)=\bar u(-x)$ and $w:=u*v$. Notice

$$w(0)=(u*v)(0)=\int_{\mathbb{R}^d}u(x)v(-x)=\int_{\mathbb{R}^d}u(x)\bar u(x)=\|u\|_{L^2}$$

By lemma ①, $w\in L^1(\mathbb{R}^d)\cap C(\mathbb{R}^d)$. Notice

$$\bar{\hat u}=\frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}e^{ix\cdot y}\bar u(x)dx=\frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}e^{-ix\cdot y}\bar u(-x)dx=\hat v$$

Therefore, $\hat w=(2\pi)^{d/2}\hat u\hat v=(2\pi)^{d/2}|\hat u|^2$. Since $w\in C(\mathbb{R}^d)$, by lemma ②,

$$(2\pi)^{d/2}w(0)=\lim_{\varepsilon\to0^+}\frac{1}{(2\varepsilon)^{d/2}}\int_{\mathbb{R}^d}w(x)e^{\frac{-|x|^2}{4\varepsilon}}dx$$

For any function $f,g\in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$, $\int_{\mathbb{R}^d}f(x)\hat g(x)dx=\int_{\mathbb{R}^d}\hat f(x)g(x)dx$

$$\therefore\underbrace{\lim_{\varepsilon\to0^+}\frac{1}{(2\varepsilon)^{d/2}}\int_{\mathbb{R}^d}w(x)e^{\frac{-|x|^2}{4\varepsilon}}}_{(2\pi)^{d/2}w(0)}=\lim_{\varepsilon\to0^+}\int_{\mathbb{R}^d}\hat w(x)e^{-\varepsilon|x|^2}$$

By dominated Convergence theorem, $(2\pi)^{d/2}w(0)=\int_{\mathbb{R}^d}\hat w(x)dx$

so $w(0)=\int_{\mathbb{R}^d}|\hat u(x)|^2=\|\hat u\|_{L^2}$

Now suppose $u\in L^2(\mathbb{R}^d)$. Fix $R>0$ and define

$$u_k=\mathbb{1}_{\overline{B_{kR}(0)}}u\quad\text{for }k=1,2,\dots$$

clearly $u_k\in L^2(\mathbb{R}^d)$ and by Cauchy Schwartz, $u_k\in L^1(\mathbb{R}^d)$. Furthermore $u_k\to u$ in $L^2(\mathbb{R}^d)$. Then for $k,j\in\mathbb{N}$,

$$\|u_k-u_j\|_{L^2}=\|\widehat{u_k-u_j}\|_{L^2}=\|\hat u_k-\hat u_j\|_{L^2}$$

Since $\{u_k\}_{k=1}^\infty$ is Cauchy in L^2, $\{\hat u_k\}_{k=1}^\infty$ is Cauchy in L^2. Therefore, there exists a limiting function $v\in L^2(\mathbb{R}^d)$. Define $\hat u:=v_k$

Theorem: For $u,v\in L^2(\mathbb{R}^d)$ the following holds:

(i) $\langle u,v\rangle_{L^2}=\langle\hat u,\hat v\rangle_{L^2}$

(ii) $\widehat{\partial^\alpha u}=(ix)^\alpha\hat u$ for each multi-index α s.t. $\partial^\alpha u\in L^2(\mathbb{R}^d)$

(iii) $(\widehat{u*v})=(2\pi)^{d/2}\hat u\hat v$

(iv) $u=\check{(\hat u)}$

Theorem: Let $k\in\mathbb{N}$ and $u\in L^2(\mathbb{R}^d)$ then

(i) $u\in H^k(\mathbb{R}^d)$ iff $(1+\|x\|_2^k)\hat u\in L^2(\mathbb{R}^d)$

(ii) There exists a constant $c>0$ s.t.

$$\frac1c\|u\|_{H^k}\le\|(1+\|x\|_2^k)\hat u\|_{L^2}\le c\|u\|_{H^k}$$

Definition: Let $0<s<\infty$ then define the fractional Sobolev space:

$$H^s(\mathbb{R}^d):=\{u\in L^2(\mathbb{R}^d):\|u\|_{H^s}<\infty\}$$

where for $s\in(0,\infty)\setminus\mathbb{N}$,

$$\|u\|_{H^s}=\|(1+\|x\|_2^s)\hat u\|_{L^2}$$

10/29 L9 (Continuous Embedding)

$S=\{s:\mathbb{N}\to\mathbb{R}\}$ sequences $\quad F=\{f:D\to\mathbb{R}\}$ functions

Today's question: Let X,Y be Banach spaces ($\subset S$ or F) when can we say $X\subset Y$?

Definition: X,Y Banach Spaces. X is continuously embedded in Y if $\exists\,C>0:\forall u\in X$ $\|u\|_Y\le C\|u\|_X$

Only relevant in infinite dimensional spaces because all norms are equivalent in finite dimensions

Picture (F)

Example:

$\ell^\infty=\{u\in S:\|u\|_{\ell^\infty}:=\sup_{j\in\mathbb{N}}|u_j|<\infty\}$

$c_0=\{u\in\ell^\infty:u_j\to0\text{ as }j\to\infty\}$

$c_0\subset\ell^\infty$ … $\|u\|_{c_0}=\|u\|_{\ell^\infty}$ $(C\equiv1)$

c_0 is continuously embedded in ℓ^∞

Definition: If $X\subseteq Y$ the *inclusion map* $i:X\to Y$ is defined by $iu=u$ $\forall u\in X$

Lemma: X is continuously embedded in Y iff $i\in\mathcal{L}(X,Y)$

Lemma: If X & Y are continuously embedded in one another (we say they are *equivalent*), then the norms $\|\cdot\|_X,\|\cdot\|_Y$ are equivalent norms.

Theorem: ℓ^r is continuously embedded in ℓ^s for every $1\le r\le s\le\infty$ & $\|u\|_{\ell^s}\le\|u\|_{\ell^r}$

Proof: $\max_{1\le j\le J}|u_j|\le(\sum_{j=1}^J|u_j|^r)^{1/r}\le(\sum_{j=1}^\infty|u_j|^r)^{1/r}$

① $r<s=\infty$:

$\therefore\sup_{j\in\mathbb{N}}|u_j|\le(\sum_{j=1}^\infty|u_j|^r)^{1/r}\implies\|u\|_{\ell^\infty}\le\|u\|_{\ell^r}$

② $r<s<\infty$: Let $u\in\ell^r$. Define $v\in S$ by $v_j:=\frac{u_j}{\|u\|_{\ell^r}}$

$$\sup_j|v_j|=\|v\|_{\ell^\infty}=\frac{\|u\|_{\ell^\infty}}{\|u\|_{\ell^r}}\le1\implies\|v\|_{\ell^\infty}\le1$$

…

ᴬLᵃ cont'd:

$$\|v\|_{\ell^s}^s = \sum_{j=1}^{\infty}|v_j|^s = \sum_{j=1}^{\infty}|v_j|^r|v_j|^{s-r} \le \sum_{j=1}^{\infty}|v_j|^r = \|v\|_{\ell^r}^r$$

But $\|v\|_{\ell^r}^r = \|u\|_{\ell^r}^r \big/ \|u\|_{\ell^r}^r = 1 \quad \therefore \|v\|_{\ell^s}^s \le 1$

$\Rightarrow \dfrac{\|u\|_{\ell^s}^s}{\|u\|_{\ell^r}^s} \le 1$ required result.

Example:

$L^p(\mathbb{R}^d;\mathbb{R}) = \{u:\mathbb{R}^d\to\mathbb{R} \mid \int_{\mathbb{R}^d}|u(x)|^p dx < \infty\}$

$H^k(\mathbb{R}^d;\mathbb{R}) = \{u:\mathbb{R}^d\to\mathbb{R}$ with weak derivatives of order up to k in $L^2(\mathbb{R}^d;\mathbb{R})\}$.

Theorem: H^K is continuously embedded into L^∞ if $k > \frac{d}{2}$

Fourier review: $\underbrace{\hat{u}(\xi)}_{(Fu)(\xi)} = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} e^{-i\langle\xi,x\rangle}u(x)\,dx$

$$(F^{-1}\hat{u})(x) = u(x) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} e^{i\langle\xi,x\rangle}\hat{u}(\xi)\,d\xi$$

$$\|u\|_{H^k}^2 \underset{\ge c_2}{\le c_1}\int(1-|\xi|^2)^k|\hat{u}(\xi)|^2 d\xi$$

Proof: Want c: $\|u\|_{L^\infty} \le c\|u\|_{H^k}$

With some work: $\|u\|_{L^\infty} \le \int|\hat{u}(\xi)|\,d\xi = \int\frac{(1+|\xi|^2)^{k/2}}{(1+|\xi|^2)^{k/2}}|\hat{u}(\xi)|\,d\xi$

$$\le \underbrace{\left(\int\frac{1}{(1+|\xi|^2)^k}d\xi\right)^{1/2}}_{<\infty \text{ if } k>d/2}\underbrace{\left(\int(1+|\xi|^2)^k|\hat{u}(\xi)|^2 d\xi\right)^{1/2}}_{\le c\|u\|_{H^k}}$$

Theorem: H^k is continuously embedded into L^p if $k \le \frac{d}{2}$ and $p \in [2, 2d/(d-2k))$

[~~Holders~~ "Believe me": $\|u\|_{L^p} \le C\|\hat{u}\|_{L^q}$, $p^{-1}+q^{-1}=1 \quad 1\le q<2$]

Proof: $\left(\frac{2}{q}\right)^{-1} + \left(\frac{2}{2-q}\right)^{-1} = 1 \quad 1<q<2$

Hölder: $\|u\|_{L^p}^q \le C_2\int_{\mathbb{R}^d}\frac{(1-|\xi|^2)^{kq/2}}{(1-|\xi|^2)^{kq/2}}|\hat{u}(\xi)|^q d\xi$

[Now using $\int ab \le (\int a^\alpha)^{1/\alpha}(\int b^\beta)^{1/\beta}$ (Holder) α,β conjugates]

$$\le C_3\underbrace{\left(\int_{\mathbb{R}^d}\frac{d\xi}{(1+|\xi|^2)^{kq/2\cdot\frac{2}{2-q}}}\right)^{\frac{2-q}{2}}}_{<\infty \text{ iff } \frac{2kq}{2-q}>d}\cdot\underbrace{\left(\int_{\mathbb{R}^d}(1+|\xi|^2)^k|\hat{u}(\xi)|^2 d\xi\right)^{q/2}}_{\le\|u\|_{H^k}^q}$$

$\leadsto p\,\&\,q$ are conjugate exponents

Example: $D\subset\mathbb{R}^d$ bounded open set

$\varphi_j: D\to\mathbb{R}$ $\qquad \sim \lambda_j \asymp j^{2/d}$ for reasonable domains

$-\Delta\varphi_j = \lambda_j\varphi_j,\ x\in D$ $\qquad \langle\varphi_j,\varphi_k\rangle_{L^2(D;\mathbb{R})} = \delta_{jk}$

$\varphi_j = 0 \quad x\in\partial D$ $\qquad \{\varphi_j\}_{j\in\mathbb{N}}$ form Orth.N basis for $L^2(D;\mathbb{R})$

$u\in L^2(D;\mathbb{R})$ then

$u = \sum_{j\in\mathbb{N}} v_j\varphi_j,\ v_j := \langle u,\varphi_j\rangle_{L^2(D;\mathbb{R})}$ $\quad \Big|\ \|u\|_{L^2(D;\mathbb{R})}^2 = \|v\|_{\ell^2(\mathbb{N};\mathbb{R})}^2$ [Parseval]

$u\in H^1(D;\mathbb{R})$ then

$$\|u\|_{H^1}^2 = \|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2 = \langle-\Delta u,u\rangle_{L^2} + \|u\|_{L^2}^2 = \sum_{j=1}^{\infty}\underbrace{(1+\lambda_j)}_{\asymp j^{2/d}}v_j^2 = \|v\|_{\ell^2 w(\mathbb{N};\mathbb{R})}^2$$

10/31 L10: Compactness

$U\subseteq V$

$(U,\|\cdot\|_U)$, $(V,\|\cdot\|_V)$ Banach

Continuous Embedding:

$\exists c>0: \forall u\in U \quad \|u\|_V \le c\|u\|_U$

compact embedding:

$\{u^{(n)}\}$ in B bounded in $U \Rightarrow \{u^{(n_j)}\}$ (subsequence) with limit in V

$\hookrightarrow U$ is compactly embedded in V

Compact Set: $(X,\|\cdot\|)$ Banach space

- $S\subset X$ is (sequentially) compact if $\{u^{(n)}\}_{n\in\mathbb{N}}$ in S contains $\{u^{(n_j)}\}_{j\in\mathbb{N}}$ with $\|u^{(n_j)}-u^*\|_X\to 0$ as $j\to\infty$ and $u^*\in S$
- Relatively (sequentially) compact if above except $u^*\in X$

Example: $X=\mathbb{R}^n$, $x=(x_1,\dots,x_n)^T$

- $S=\{-1\le x_j\le 1,\ j=1,\dots,n\}$ sequentially compact
- $S=\{-1<x_j<1,\ j=1,\dots,n\}$ relatively sequentially compact

Aside [Compact operator: $(U,\|\cdot\|_U)$, $(V,\|\cdot\|_V)$ Banach
Definition: $L:V\to U$ is compact if LB is relatively compact whenever B is bounded
compact embedding:
Definition: U is compactly embedded in V if $i:U\to V$ is a compact operator]

Important Example: $\ell^2 = \ell^2(\mathbb{N};\mathbb{R})$

$\mathcal{H}^s = \{u\in\ell^2 \mid \|u\|_{\mathcal{H}^s} := (\sum_{j\in\mathbb{N}} j^{2s}|u_j|^2)^{1/2} < \infty\}$

$\mathcal{H}^0 \equiv \ell^2$. We show that $\{u^{(n)}\}$ bounded in $\mathcal{H}^s$, any $s>0$, then there is a convergent subsequence $\{u^{(n_j)}\}$ in ℓ^2.

i.e. $\mathcal{H}^s$ is compactly embedded in ℓ^2 for any $s>0$.

Proof: $M<\infty$. Assume s is fixed and positive

Let $\{u^{(n)}\}$ lie in $B=\{u\in\ell^2 \mid \|u\|_{\mathcal{H}^s}^2 \le M\}$

Let $u^{(n)} = (C_1^{(n)}, C_2^{(n)},\dots) = \{C_k^{(n)}\}_{k\in\mathbb{N}}$

① $\sup_{n\in\mathbb{N}}\sum_{k=1}^{\infty}k^{2s}|C_k^{(n)}|^2 \le M \Rightarrow \sup_{n\in\mathbb{N}}|C_k^{(n)}| \le M^{1/2}$ for each $k\in\mathbb{N}$

By compactness in $\mathbb{R}$ implies $\exists C_1^*\ \&\ n_1(j)\to\infty$ as $j\to\infty$ s.t. $C_1^{(n_1(j))}\to C_1^*$

$\therefore \sup_{j\in\mathbb{N}}|C_2^{(n_1(j))}| \le M^{1/2}$

Now take $n_2(j)\to\infty$ as $j\to\infty$ (nested within $n_1(j)$). $\exists C_2^*$:

$(C_1^{(n_2(j))}, C_2^{(n_2(j))}) \to (C_1^*, C_2^*)$

Recurse:

$(C_1^{n_k(j)}, C_2^{n_k(j)},\dots, C_k^{n_k(j)}) \to (C_1^*, C_2^*,\dots C_k^*)$

Note: $\{n_k(j)\}$ is a subsequence of $\{n_1(j)\}$ which is a subseq. of $\{n\}$

$n_k(j)\to\infty$ as $j\to\infty$

Define $u^* = (C_1^*, C_2^*,\dots, C_k^*,\dots)$ and $\tilde{u}^{(j)} = u^{(n_j(j))}$

$n_j(j) \ge n_1(j)$ based on our definitions

$\downarrow \infty \quad \downarrow \infty$ as $j\to\infty$ $\quad\leadsto$ Remains to show: $\|\tilde{u}^{(j)}-u^*\|_{\ell^2}\to 0$ as $j\to\infty$

For each ℓ $\sum_{k=1}^{\ell}k^{2s}|C_k^*|^2 \le M$ From ① using convergence along $n_\ell(j)$, $j\to\infty$

& so $\sum_{k=1}^{\infty}k^{2s}|C_k^*|^2 \le M$ ② $[\Rightarrow u^*\in\ell^2]$

$\tilde{u}^{(j)} = (\tilde{C}_1^{(j)}, \tilde{C}_2^{(j)},\dots) = \{\tilde{C}_k^{(j)}\}_{k\in\mathbb{N}}$

$$\sup_j\sum_{k=1}^{\infty}k^{2s}|\tilde{C}_k^{(j)}-C_k^*|^2 \le \sup_j\sum_{k=1}^{\infty}\big(\underbrace{2k^{2s}|\tilde{C}_k^{(j)}|^2}_{\le 2M \text{ by ①}} + \underbrace{2k^{2s}|C_k^*|^2}_{\le 2M \text{ by ②}}\big) \le 4M$$

$$\|\tilde{u}^{(j)}-u^*\|_{\ell^2}^2 = \sum_{k=1}^{\infty}|\tilde{C}_k^{(j)}-C_k^*|^2 = \sum_{k=1}^{\ell}|\tilde{C}_k^{(j)}-C_k^*|^2 + \sum_{k=\ell+1}^{\infty}|\tilde{C}_k^{(j)}-C_k^*|^2$$

$$\le \sum_{k=1}^{\ell}|\tilde{C}_k^{(j)}-C_k^*|^2 + \frac{1}{\ell^{2s}}\sum_{k=\ell+1}^{\infty}k^{2s}|\tilde{C}_k^{(j)}-C_k^*|^2 \le \sum_{k=1}^{\ell}|\tilde{C}_k^{(j)}-C_k^*|^2 + \frac{1}{\ell^{2s}}\sum_{k=1}^{\infty}k^{2s}|\tilde{C}_k^{(j)}-C_k^*|$$

$$\|\tilde{u}^{(j)}-u^*\|_{\ell^2}^2 \le \sum_{k=1}^{\ell}|\tilde{C}_k^{(j)}-C_k^*|^2 + 4M/\ell^{2s}$$

$(\tilde{C}_1^{(j)},\dots\tilde{C}_\ell^{(j)}) \to (C_1^*,\dots,C_\ell^*)$ in $\mathbb{R}^\ell$ ③

10/31 <u>L10 cont'd:</u>

NTS for any $\varepsilon > 0$ $\exists J = J(\varepsilon)$: $\|\tilde{u}^{(j)} - u^*\|_{\ell^2}^2 < \varepsilon$

choose ℓ : $4M/\ell^{2s} < \varepsilon^2/2$. ④ For this ℓ choose J:

⑤ $\sum_{k=1}^{\ell} |\tilde{c}_k^{(j)} - c_k^*|^2 < \varepsilon^2/2$ $\forall j \geq J$ (possible by ③)

thus, for any $\varepsilon > 0$, $\|\tilde{u}^{(j)} - u^*\|_{\ell^2}^2 < \varepsilon^2$ $\forall j \geq J$

J depends on (λ, ε) & ℓ depends on ε ∎

<u>Example:</u> $I = (0,1)$; $du/dt = v$, $u(0) = 0$

$\mathcal{V} = L^2(I;\mathbb{R})$ and $\mathcal{U} = H^1(I;\mathbb{R})$

Fact: $\mathcal{U}$ is compactly embedded into $\mathcal{V}$

$u(t) = \int_0^t v(s)\, ds$ $t \in I$

$|u(t)| \leq \int_0^t |v(s)|\, ds \leq \left(\int_0^t |v(s)|^2 ds\right)^{1/2} \left(\int_0^t 1\, ds\right)^{1/2}$

$\leq \left(\int_0^1 |v(s)|^2 ds\right)^{1/2} = \|v\|_{L^2}$

$\|u\|_{L^\infty} = \sup_{t \in I} |u(t)| \leq \|v\|_{L^2}$. $\Rightarrow L \in \mathcal{L}(L^2; L^\infty)$

$\|L\|_{\mathcal{L}(L^2;L^\infty)} = \sup_v \frac{\|Lv\|_{L^\infty}}{\|v\|_{L^2}} = \sup_v \frac{\|u\|_{L^\infty}}{\|v\|_{L^2}} \leq 1$

$\|u\|_{L^2}^2 = \int_0^1 |u(t)|^2 dt \leq \int_0^1 \|u\|_{L^\infty}^2 dt \leq \|v\|_{L^2}^2$

$\|L\|_{\mathcal{L}(L^2;L^2)} = \sup_v \frac{\|Lv\|_{L^2}}{\|v\|_{L^2}} = \sup_v \frac{\|u\|_{L^2}}{\|v\|_{L^2}} \leq 1$

$\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|\frac{du}{dt}\|_{L^2}^2 = \|u\|_{L^2}^2 + \|v\|_{L^2}^2 \leq 2\|v\|_{L^2}^2$.

$\|L\|_{\mathcal{L}(L^2;H^1)}^2 = \sup_v \frac{\|Lv\|_{H^1}^2}{\|v\|_{L^2}^2} = \sup_v \frac{\|u\|_{H^1}^2}{\|v\|_{L^2}^2} \leq \sup_v \frac{2\|v\|_{L^2}^2}{\|v\|_{L^2}^2} = 2$

$\boxed{\|L\|_{\mathcal{L}(L^2,H^1)} \leq \sqrt{2}}$ $\therefore$ L is a compact operator.

11/1 Recitation: Review of Duality:

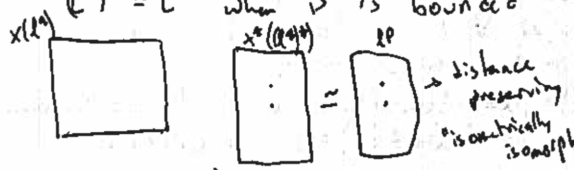
$(V, \|\cdot\|)$ over $\mathbb{K}$ $\to$ $V^* := \mathcal{L}(V, \mathbb{K})$

<u>Example</u> (from class)

$(\ell^q)^* = \ell^p$ $\forall q \in [1,\infty)$, $\frac{1}{p} + \frac{1}{q} = 1$; $(\ell^\infty)^* \supset \ell^1$

<u>Ex</u>: $(L^q(D;\mathbb{R}))^* \simeq L^p(D;\mathbb{R})$ — 〃 —

$(L^1)^* \simeq L^\infty$ when D is bounded



<u>Density:</u> $(X, \|\cdot\|)$ NVS $A \subset X$

<u>Definition:</u> A is <u>dense</u> in X if for any $v \in X$ and every $\varepsilon > 0$ $\exists$ $a \in A \Rightarrow \|v - a\| < \varepsilon$

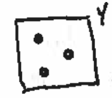
Ex: $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$

Definition: If there exists A that is dense and countable then X is <u>separable</u>.

Remark: If X (NVS) is Banach and A is dense in X then $X = \bar{A}$ (closure of A)

<u>Theorem:</u> ℓ^p is separable $\forall p \in [1,\infty)$; ℓ^∞ is not separable

Same is true for $L^p(D;\mathbb{R})$



$\bar{Y}$ is closure

<u>Completion:</u> $(X, \|\cdot\|)$ Banach $Y \subset X$

<u>Definition:</u> Completion of Y in X is

$\bar{Y} := \{u \in X \mid \exists \{u_n\} \subset Y \text{ s.t. } \|u - u_n\| \to 0\}$ — $u = \lim_{n\to\infty} u_n$

Remark: Q: closure vs. completeness?

A: · for closure, need ambient space → "Y is closed in X"

· for completion, only need $\|\cdot\|$ → "Y is complete"

Ex: · Completion of $\mathbb{Q}$ is $\mathbb{R}$ with mod

· Closure of $\mathbb{Q}$ in $\mathbb{Q}$ is $\mathbb{Q}$

Remark: · If Y is a subspace then $\bar{Y}$ is Banach

· If Y is dense $\Rightarrow \bar{Y} = X$

<u>Definition:</u> $H_0^k(D;\mathbb{R})$ is the completion of $C_c^\infty(D;\mathbb{R})$ in $H^k(D;\mathbb{R})$

- H_0^k is a Hilbert space with H^k inner product

<u>Theorem:</u> $H_0^1(D;\mathbb{R})$ is a Hilbert space

$\langle u, v \rangle_{H_0^1} := \langle \nabla u, \nabla v \rangle_{L^2}$

<u>Lemma:</u> [Poincaré inequality] $\|u\|_{L^2} \leq C\|\nabla u\|_{L^2}$ $\forall u \in H_0^1(D;\mathbb{R})$

C is independent of u

<u>Schauder Basis:</u> $(X, \|\cdot\|)$ Banach over $\mathbb{R}$

<u>Definition:</u> A countable collection of elements $\{\varphi_n\}_{n\in\mathbb{N}} \subset X$ is called a <u>schauder basis</u> if for every element $x \in X$ there is a unique sequence $\{v_n\} \subset \mathbb{R}$ s.t.

$$\lim_{n\to\infty} \left\| x - \sum_{j=1}^{n} v_j \varphi_j \right\| = 0$$

Ex: Fourier basis for $L^p(D;\mathbb{R})$ $p \in (1,\infty)$

<u>Definition:</u> if X is Hilbert

1) orthogonal basis (*) $\langle \varphi_n, \varphi_k \rangle = 0$ $\forall n \neq k$

2) orthonormal basis (*) & $\langle \varphi_n, \varphi_n \rangle = 1$

<u>Remark:</u> Schauder basis exists $\Longrightarrow$ separability

Ex: ℓ^p has Schauder basis for $p \in [1,\infty)$, ℓ^∞ does not

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11/5 <u>L11: CH 10 (ORTHOGONALITY)</u>

$(H, \langle\cdot,\cdot\rangle, \|\cdot\|)$ Hilbert space

<u>Running Example:</u> $H = \ell^2(\mathbb{N};\mathbb{R})$; $\langle a, b \rangle = \sum_{j\in\mathbb{N}} a_j b_j$; $\|a\| = \Sigma a$

<u>Lemma (Parallelogram Rule):</u>

$\forall v, w \in H$ $\|v+w\|^2 + \|v-w\|^2 = 2(\|v\|^2 + \|w\|^2)$

<u>Example:</u>

$(a_j + b_j)^2 + (a_j - b_j)^2 = 2(a_j^2 + b_j^2)$

<u>Definition:</u> $U \subseteq V$, V vector space. U is convex if $\forall a, b \in U$, $\lambda \in [0,1]$ $\lambda a + (1-\lambda) b \in U$.

<u>Definition:</u> A (linear, vector) <u>subspace</u> $V' \subseteq V$, V a vector space is a vector space V' contained in V

11/5 L 11 cont'd:

Theorem: Let U be a closed convex set in H & $h \in H$. Then $\exists!$ (unique) $h' \in U$ s.t. $\|h-h'\| = \inf_{a \in U} \|h-a\|$

Example: Fix $J \in \mathbb{N}$. Then $U = \{u \in \ell^2(\mathbb{N};\mathbb{R}) \mid u_j = 0 \ \forall j \geq J+1\}$

Convex? Yes. If $a, b \in U$ then $a_j = b_j = 0 \ \forall j \geq J+1$

$\therefore (1-\lambda)a_j + \lambda b_j = 0 \ \forall j \geq J+1$

$\therefore (1-\lambda)a + \lambda b \in U$.

Closed? Yes. NTS that if $\{u^{(n)}\}_{n \in \mathbb{N}}, u^{(n)} \in \ell^2(\mathbb{N};\mathbb{R})$ & $u^{(n)} \to u^*$ in $\ell^2(\mathbb{N};\mathbb{R})$ then $u^* \in U$.

$u^{(n)} = \{u_1^{(n)}, u_2^{(n)}, u_3^{(n)}, \ldots\}$ $\quad u_j^{(n)} = 0 \quad j \geq J+1, n \in \mathbb{N}$

$u^* = \{u_1^*, u_2^*, u_3^*, \ldots\}$ $\quad u_j^* = 0 \quad j \geq J+1$

We have $\|u^{(n)} - u^*\|^2 \to 0$ as $n \to \infty$ $\forall \varepsilon > 0 \ \exists N = N(\varepsilon)$:

$\forall n \geq N \ \sum_{j \in \mathbb{N}} |u_j^{(n)} - u_j^*|^2 \leq \varepsilon$

$\forall n \geq N \ \sum_{j=1}^{J} |u_j^{(n)} - u_j^*|^2 + \sum_{j \geq J+1} |u_j^*|^2 \leq \varepsilon$

$\forall n \geq N \ \forall j \geq J+1 \ |u_j^*| \leq \varepsilon$

$\therefore u_j^* = 0 \ \forall j \geq J+1$ since $\varepsilon > 0$ is arbitrary.

Example: Closest point in U

$h = (h_1, h_2, h_3, \ldots, h_J, h_{J+1}, h_{J+2}, \ldots)$ $\quad h \in \ell^2(\mathbb{N};\mathbb{R})$

$h' = (h_1, h_2, h_3, \ldots, h_J, 0, 0, \ldots)$

Let $a \in U$, $a \neq h'$ $\therefore a = (a_1, \ldots, a_J, 0, 0 \ldots)$

$(a_1, \ldots, a_J) \neq (h_1, \ldots, h_J)$

$\|h-a\|^2 = \sum_{j=1}^{J} (h_j - a_j)^2 + \sum_{j \geq J+1} h_j^2$

$\|h-h'\|^2 = 0 \quad + \quad -\text{"}-$

$\|h-h'\|^2 \leq \|h-a\|^2$ with equality only if $a \equiv h'$.

Definition: $v, w \in H$ are orthogonal if $\langle v, w \rangle = 0$.

Definition: For a subspace $U \subset H$ we define the orthogonal complement $U^\perp$ by

$U^\perp = \{v \in H : \langle v, w \rangle = 0 \ \forall w \in U\}$

Lemma: The orthogonal complement satisfies

- For subspace $U \subset H$, $U^\perp$ is a closed subspace in H
- $U \cap U^\perp = 0$
- If U is a closed subspace then $(U^\perp)^\perp = U$

Example: $H = \ell^2 = \ell^2(\mathbb{N};\mathbb{R})$. $J \in \mathbb{N}$ fixed

$U = \{u \in \ell^2 \mid u_j = 0, \ j \geq J+1\}$. U is closed (see earlier). U is also a subspace of ℓ^2.

$U^\perp = \{v \in \ell^2 \mid \langle v, w \rangle = 0 \quad \forall w \in U\}$

$= \{v \in \ell^2 \mid \sum_{j=1}^{J} v_j w_j = 0 \ \forall (w_1, \ldots, w_J) \in \mathbb{R}^J\}$

$= \{v \in \ell^2 \mid (v_1, \ldots, v_J) \equiv 0\}$

Then: $U^\perp$ is closed; $U^\perp \cap U = 0$; $(U^\perp)^\perp$ is U.

Theorem: If U is a closed subspace in H, $h \in H$ & $h' \in U$ is the closest point to h in U then $(h-h') \in U^\perp$. Thus $H = U \oplus U^\perp$: any $h \in H$ can be written uniquely as $h = p + q$, $p \in U$, $q \in U^\perp$

Content: $h = \underbrace{h'}_{p \in U} + \underbrace{(h-h')}_{q \in U^\perp}$

Example: $H = \ell^2(\mathbb{N};\mathbb{R})$, U as above

$h = (h_1, h_2, h_3, \ldots) \in \ell^2(\mathbb{N};\mathbb{R})$

$p = (h_1, \ldots, h_J, 0, \ldots)$

$q = (0, \ldots 0, h_{J+1}, \ldots)$

Theorem: Let U be a closed subspace in H and define $P: H \to H$ by $Ph = p \ \forall h \in H$. (p is the closest point to h in U). Then $P \in \mathcal{L}(H,H)$ (a), $P^2 = P$ (b) & $\|P\|_{\mathcal{L}(H,H)} = 1$ (c)

Proof: a) $\alpha, \beta \in \mathbb{R}$, $v, w \in H$

$\alpha v + \beta w = P(\alpha v + \beta w) + q$, $q \in U^\perp$

$v = Pv + q_1$, $q_1 \in U^\perp$

$w = Pw + q_2$, $q_2 \in U^\perp$

$P(\alpha v + \beta w) + q = \alpha(Pv + q_1) + \beta(Pw + q_2)$

$\underbrace{P(\alpha v + \beta w)}_{\in U} + \underbrace{q}_{\in U^\perp} = \underbrace{\alpha Pv + \beta Pw}_{\in U} + \underbrace{\alpha q_1 + \beta q_2}_{\in U^\perp}$

By uniqueness $P(\alpha v + \beta w) = \alpha Pv + \beta Pw$ i.e. P is linear

b) $h = p + q$, $p \in U$, $q \in U^\perp$, $Ph = p$, $P^2 h = Pp$. Clearly since $p \in U$ $Pp = p$, i.e. $\|p - Pp\| = \inf_{a \in U} \|p - a\|$

$= 0$ if $Pp = p$ $\Rightarrow P^2 h = p = Ph \quad \forall h \in H \ \therefore P^2 = P$

c) $h - p \perp p$. $\to \|h\|^2 = \|p + h - p\|^2 = \|p\|^2 + \cancel{2\langle p, h-p \rangle}^{0} + \|h-p\|^2$

$= \|p\|^2 + \|h-p\|^2 \geq \|p\|^2 \to \|h\|^2 \geq \|Ph\|^2$

$\|P\|_{\mathcal{L}(H,H)} = \sup_{h \in H} \frac{\|Ph\|}{\|h\|} \leq \sup_{h \in H} \frac{\|h\|}{\|h\|} = 1$

If $h \in U$ then $h - p = 0$ & $\|h\|^2 = \|Ph\|^2$ & so $\sup_{h \in H} \frac{\|Ph\|}{\|h\|} = 1$

Example: $H = \ell^2(\mathbb{N};\mathbb{R})$, U as before

$h = (h_1, h_2, \ldots, h_J, h_{J+1}, \ldots)$

$Ph = (h_1, h_2, \ldots h_J, 0, \ldots)$

$P^2 h = (h_1, h_2, \ldots, h_J, 0, \ldots)$

L 12 Background (Ch 3): $D \subset \mathbb{R}^d$ bounded open

$-\nabla \cdot (a \nabla p) = f$, $x \in D$; $p = 0$ $x \in \partial D$

$f \in L^2(D;\mathbb{R})$, $a \in L^\infty(D;\mathbb{R})$

$-\int_D \nabla \cdot (a \nabla p) q \, dx = \int_D f q \, dx \xRightarrow{\text{IBP}} + \int_D a \nabla p \nabla q \, dx = \int_D f q \, dx$

Find $p \in V$: $B(p,q) = r(q) \quad \forall q \in V$

$V = H_0^1(D;\mathbb{R})$

11/7 Reisz Representation Theorem:

$(H, \langle \cdot, \cdot \rangle, \|\cdot\|)$; $H = U \oplus U^\perp$

$f \in \mathcal{L}(H;\mathbb{R}) \to \exists w \in H : \forall v \in H \ f(v) = \langle w, v \rangle$

Theorem: For every $f \in H^* = \mathcal{L}(H;\mathbb{R})$ there exists a unique $w \in H$: $f(v) = \langle w, v \rangle \ \forall v \in H$. Furthermore $\|w\|_H = \|f\|_{H^*}$

Proof: Idea: (a) $\exists w$ (b) $!w$ (c) $\|\cdot\|$

a) $K = \ker(f) = \{u \in H : f(u) = 0\}$. If $u, v \in K$ & $\alpha, \beta \in \mathbb{R}$ then $f(\alpha u + \beta v) = \alpha f(u) + \beta f(v) = 0$ i.e. $\alpha u + \beta v \in K$ $\Big\}$ K is a linear subspace

$\therefore K$ is a subspace of H.

Let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence in K & $u_n \to u$ in H. Then $u \in K$. $\Big\}$ K is closed

For any $\varepsilon > 0$ $\exists N = N(\varepsilon)$: $\|u_n - u\| < \varepsilon \ \forall n \geq N$.

$|f(u)| = |f(u) - f(u_n)| = |f(u - u_n)|$

$|f(u)| \leq \|f\|_{H^*} \|u - u_n\|$. $\forall \varepsilon > 0 \ \exists N = N(\varepsilon) : |f(u)| \leq \|f\|_{H^*} \varepsilon \ \forall n \geq N(\varepsilon)$

Since $\varepsilon > 0$ is arbitrary, $|f(u)| = 0$.

$H = K \oplus K^\perp$ (from 11/5). If $f \equiv 0$ then $K \equiv H$, $K^\perp = \{0\}$. In this case $w = 0$. Now assume $f \neq 0$. Then $K \neq H$, $K^\perp \neq \{0\}$. In fact $K^\perp$ is one-dimensional: if $u, v \in K^\perp$ then u & v are proportional to one another. ~~If $u, v \in K$ & $\alpha, \beta \in \mathbb{R}$ then $f(\alpha u + \beta v) = \alpha f(u) + \beta f(v) = 0$. i.e. $\alpha u + \beta v$~~. $f(f(u)v - f(v)u) =$ $f(u) f(v) - f(v) f(u) = 0$ i.e. $f(u)v - f(v)u \in K$. Since $u, v \in K^\perp$, $f(u)v - f(v)u \in K^\perp$ i.e. $f(u)v = f(v)u$

11/7 <u>L 12 cont'd:</u>

Choose $z \in K^\perp$ without loss of generality $\|z\|=1$. Using $H = K \oplus K^\perp$ it follows that for any $v \in H$, $\exists!$ $u \in K$ and $\alpha \in \mathbb{R}$:

$$v = \underbrace{\alpha z}_{\in K^\perp} + \underbrace{u}_{\in K} \; ; \; \langle z, v\rangle = \alpha \langle z, z\rangle + \langle z, u\rangle = \alpha$$

$$f(v) = f(\alpha z + u) = \alpha f(z) + \cancel{f(u)} = f(z)\langle z, v\rangle = \underbrace{\langle f(z) z, v\rangle}_{:= w}$$

b) If $\exists$ $w, \tilde{w}$: $\forall v \in H$ $f(v) = \langle w, v\rangle = \langle \tilde{w}, v\rangle$. Then $\langle w - \tilde{w}, v\rangle = 0$ $\forall v \in H$. $\therefore$ $w - \tilde{w} \in H^\perp = \{0\}$ i.e. $w = \tilde{w}$

c) $f(v) = \langle w, v\rangle \to |f(v)| \le \|w\|\|v\| \to \frac{|f(v)|}{\|v\|} \le \|w\|$ $\forall v \in H$ ①

& $\frac{f(w)}{w} = \frac{\langle w, w\rangle}{\|w\|} = \frac{\|w\|^2}{\|w\|} = \|w\|$ ②

① & ② $\Rightarrow$ $\underbrace{\sup_v \frac{|f(v)|}{\|v\|}}_{\|f\|_{H^*}} = \|w\|$

$\|f\|_{H^*} = \|w\|$.

<u>THE EXAMPLE:</u> $D \subset \mathbb{R}^d$ bounded, open

$$\boxed{\begin{aligned} -\nabla\cdot(a \nabla u) &= r, \quad x \in D \\ u &= 0, \quad x \in \partial D \end{aligned}} \text{ PDE}$$

$a \in L^2(D;\mathbb{R})$; $r \in L^2(D;\mathbb{R})$; $H = H_0^1(D;\mathbb{R})$

$\boxed{\text{Find } u \in H: \; B(u,v) = f(v) \quad \forall v \in H}$ WPDE

$B(u,v) = \int_D a(x) \langle \nabla u(x), \nabla v(x)\rangle_{\mathbb{R}^d} dx$; $f(v) = \int_D r(x) v(x) dx$

Assumptions: • $a \in L^\infty(D;\mathbb{R})$, $f \in L^2(D;\mathbb{R})$

• $\exists$ $a^-, a^+ \in (0,\infty)$: $0 < a^- \le \inf_{x \in D} a(x) \le \sup_{x \in D} a(x) \le a^+ < \infty$

<u>Theorem:</u> Under assumptions, $\exists!$ $u \in H$ solving (WPDE) &

(*) $\|u\|_H \le \frac{C_p}{a_-} \|r\|_{L^2}$.

<u>Facts About $H_0^1(D;\mathbb{R})$</u>

• $H^1(D;\mathbb{R}) = \{u : D \to \mathbb{R} \mid \int_D |u(x)|^2 + |\nabla u(x)|^2 dx < \infty\}$

$\langle u, v\rangle_{H^1} = \langle u, v\rangle_{L^2} + \langle \nabla u, \nabla v\rangle_{L^2}$; $\|u\|_{H^1}^2 = \langle u, u\rangle_{H^1}$

• $H_0^1(D;\mathbb{R})$ is completion w.r.t. $H^1(D;\mathbb{R})$ norm of $C_c(D;\mathbb{R})$.

{Roughly: forces elements of $H_0^1(D;\mathbb{R})$ to be zero on D}.

◦ $C_p := \sup_{u \in H_0^1} \frac{\|u\|_{L^2}}{\|\nabla u\|_{L^2}} < \infty$ i.e. $\exists C_p : \forall u \in H_0^1$ $\|u\|_{L^2} \le C_p \|\nabla u\|_{L^2}$

If $u \in H_0^1$ then $\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2$

$$\|u\|_{H^1}^2 \begin{cases} \ge \|\nabla u\|_{L^2}^2 \\ \le (1 + C_p)\|\nabla u\|_{L^2}^2 \end{cases} \Rightarrow \text{norm equivalence}$$

• $H_0^1(D;\mathbb{R}) = \{u : D \to \mathbb{R} \mid \int |\nabla u(x)|^2 dx < \infty\}$

$\langle u, v\rangle_{H_0^1} = \langle \nabla u, \nabla v\rangle_{L^2}$, $\|u\|_{H_0^1}^2 = \|\nabla u\|_{L^2}^2$.

<u>Proof of (*):</u> Ⓐ $f \in \mathcal{L}(H;\mathbb{R})$

Ⓑ $B(u,v)$ defines an inner-product and hence norm on $H = H_0^1(D;\mathbb{R})$

Ⓒ By Reisz representation theorem Ⓐ, Ⓑ $\Rightarrow \exists!$ u solving WPDE

Ⓐ) $|f(v)| \le \|r\|_{L^2}\|v\|_{L^2} \le C_p \|r\|_{L^2} \|\nabla v\|_{L^2} = C_p \|r\|_{L^2} \|v\|_H$

$\therefore$ $\underbrace{\sup_{v \in H} \frac{|f(v)|}{\|v\|_H}}_{\|f\|_{H^*}} \le C_p \|r\|_{L^2} \Rightarrow f \in \mathcal{L}(H;\mathbb{R})$

$\|f\|_{H^*} \le C_p \|r\|_{L^2}$

Ⓑ) $B(u,v)$ is an inner-product (basic axioms)

$|B(u,v)| \le \|a\|_{L^\infty} \|u\|_H \|v\|_H \le a^+ \|u\|_H \|v\|_H$

$B(u,u) \le a^+ \|u\|_H^2$

$B(u,u) \ge a^- \int |\nabla u(x)|^2 dx = a^- \|u\|_H^2$

$\|u\|_B = B(u,u)^{1/2}$ is equivalent to $\|u\|_H$

Ⓒ True. Now show boundedness:

$v = u$ in WPDE

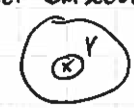
$a^- \|u\|_H^2 \le B(u,u) = f(u) \le |f(u)| \le \|f\|_{H^*} \|u\|_H$

$\le C_p \|r\|_{L^2} \|u\|_H$.

11/8 Recitation:

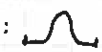
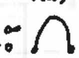
- Top $\supseteq$ Metric $\supseteq$ NVS $\supseteq$ IPS
- Banach: complete NVS
- Hilbert: complete IPS
- Sobolev spaces : $W^{k,p} \to$ Banach

  $H^k \equiv W^{k,2} \to$ Hilbert
- X, Y are NVS ; $\mathcal{L}(X,Y)$ is Banach if Y is Banach
- Dual Spaces: $X^* = \mathcal{L}(X, \mathbb{K})$
- Isometrically Isomorphic : $\exists$ linear invertible map $T: X \to Y$ that preserves the norm
- Density : $A \subset X$; $\bar{A} = X$, A is dense in X
- X is seperable : $\exists$ A which is dense & countable
- Continuous embedding, Sobolev embedding theorem
  - $H^k(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ $\quad k > \frac{d}{2}$
  - $H^k(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$ $\quad k \ge \frac{d}{2}$ $\quad p \in [2, \frac{2d}{2d-k})$
  - $\ell^r \hookrightarrow \ell^s$ $\quad 1 \le r \le s \le \infty$
- Compact embedding



bounded sequence in X

$\Rightarrow$ has convergent subsequence in Y

- C_c^∞ :  ; C_0^∞ 
- If $u \in L^2(\mathbb{R}^d)$ then $u \in H^k(\mathbb{R}^d) \Leftrightarrow (1 + |\xi|^2)^{k/2} \hat{u} \in L^2(\mathbb{R}^d)$
- $s \in \mathbb{R}^+$, $H^s(\mathbb{R}^d) := \overline{C_0^\infty(\mathbb{R}^d)}^{\|\cdot\|_{H^s}}$ where

  $\|u\|_{H^s} := \|(1 + |\cdot|^2)^{s/2} \hat{u}(\cdot)\|_{L^2}$
- H^{-k} is the dual space of H^k $\rightsquigarrow H^{-k} = (H^k)^*$

<u>Example:</u> H_0^{-1} is the dual of H_0^1, meaning $f \in H_0^{-1}(\mathbb{R}^d)$ if it's a bounded linear functional on $H_0^1(\mathbb{R}^d)$

<u>Lemma:</u> If $f \in H_0^{-1}(\mathbb{R}^d)$ then $\exists$ $f^0, f^1, \ldots, f^d \in L^2(\mathbb{R}^d)$ s.t.

$$f(v) = \int_{\mathbb{R}^d} f^0 v + \sum_{i=1}^d f^i \partial^{x_i} v \, dx \quad \forall v \in H_0^1(\mathbb{R}^d)$$

$$f(v) = \langle \tilde{f}, v\rangle \; , \; \tilde{f} = (f^0, \ldots, f^d)$$

11/9 Recitation cont'd:

Hahn-Banach Theorem: X, Banach, U subspace of X

$\hat{f}: U \to \mathbb{R}$, $\hat{f} \in U^*$, $|\hat{f}(u)| \leq M\|u\|$ $\forall u \in U$

Then $\exists$ $f \in X^*$ s.t. $f|_u = \hat{f}$ (f restricted to domain U) and $|f(u)| \leq M\|u\|$ $\forall u \in X$. In other words,

$\|f\|_{X^*} = \|\hat{f}\|_{U^*}$

Corollary: Let $u, v \in X$ if $f(u) = f(v)$ $\forall f \in X^* \Rightarrow u = v$

11.4 in LN: $(H, \langle \cdot, \cdot \rangle)$ be IPS (not necessarily Hilbert)

Let $\{e_j\}_{j=1}^n$ be an o/n set: $\langle e_i, e_j \rangle = \delta_{ij}$

$U = \text{span}\{e_j\}_{j=1}^N : \sum_{j=1}^n \alpha_j e_j$, $\alpha_j \in \mathbb{K}$

Then closest point $p \in U$ to $h \in H$ is

$p = \sum_{j=1}^n \langle h, e_j \rangle e_j$

Proof: 1) U is closed

2) $H = U \oplus U^\perp \to h = p + q$, $q \in U^\perp$

$\Rightarrow h - p \in U^\perp \Rightarrow \langle h-p, e_j \rangle = 0$ $\forall j = 1, \dots, n$

$\langle h, e_j \rangle = \langle p, e_j \rangle$ $\forall j = 1, \dots, n$

$p = \sum_k \alpha_k e_k \Rightarrow \langle p, e_j \rangle = \alpha_k$

CMS 107 $\qquad H = \ell^2(\mathbb{N};\mathbb{R})$

9/14

$H, \langle\cdot,\cdot\rangle, \|\cdot\|$; $T \in \mathcal{L}(H,H) =: B$; $\text{Rank}(T) = \text{Dim}(TH)$

<u>Definition:</u> $B_{00} = \{T \in B \mid T \text{ has finite rank}\}$

$B_0 = \{T \in B \mid T \text{ is compact}\}$

<u>Spectral Theorem:</u>

<u>Proposition:</u> If $T \in B_{00}$ has rank r and T is symmetric then $\exists \{w^{(j)}\}_{j=1}^r$, $\{\lambda^{(j)}\}_{j=1}^r$ eigenvector-eigenvalue pairs for T with $\langle w^{(j)}, w^{(k)}\rangle = \delta^{jk}$, $\lambda^{(j)} \in \mathbb{R}$ s.t. $T = \sum_{j=1}^r \lambda^{(j)} w^{(j)} \otimes w^{(j)}$

<u>Theorem:</u> If T is symmetric and $T \in B_0 \setminus B_{00}$ then $\exists \{w^{(j)}\}_{j=1}^\infty$, $\{\lambda^{(j)}\}_{j=1}^\infty$ eigenvector-eigenvalue pairs for T with $\langle w^{(j)}, w^{(k)}\rangle = \delta_{ij}$, $\lambda^{(j)} \in \mathbb{R}$ s.t. $T = \lim_{r\to\infty} T_r$ in B where $T_r = \sum_{j=1}^r \lambda^{(j)} w^{(j)} \otimes w^{(j)}$

$$\text{EVP: } \left\{ \begin{array}{l} w^{(j)}, \lambda^{(j)} \in H \times \mathbb{C}: \\ Tw^{(j)} = \lambda^{(j)} w^{(j)} \\ \|w^{(j)}\|^2 = 1 \end{array} \right\}$$

<u>Result:</u> If $T \in B$ is symmetric then $\lambda^{(j)} \in \mathbb{R}$ & $\langle w^{(j)}, w^{(k)}\rangle = \delta_{jk}$

<u>Definition:</u> $T \in B$, symmetric, is positive if $\langle Tu, u\rangle \geq 0 \ \forall u \in H$

<u>Example:</u> $H = \ell^2(\mathbb{N};\mathbb{R})$. $v = Ku$, $v_i = \sum_{j\in\mathbb{N}} K_{ij} u_j$, $K_{ij} = K_{ji}$,

$\exists s > 0: \sum_{(i,j)\in\mathbb{N}\times\mathbb{N}} (i^s K_{ij})^2 < \infty$

$\mathcal{H}^s = \{h \in H : \|h\|_{\mathcal{H}^s} := (\sum_{j\in\mathbb{N}} j^{2s} h_j^2)^{1/2} < \infty\}$, $\mathcal{H}^s$ compactly emb. in ℓ^2

Therefore, K is compact if it maps a bounded set in ℓ^2 into a bounded set in $\mathcal{H}^s$.

$\|v\|_{\mathcal{H}^s}^2 = \sum_{i\in\mathbb{N}} i^{2s} v_i^2 = \sum_{i\in\mathbb{N}} i^{2s} (\sum_{j\in\mathbb{N}} K_{ij} u_j)^2$

$\leq \sum_{i\in\mathbb{N}} i^{2s} (\sum_{j\in\mathbb{N}} K_{ij}^2)(\sum_{j\in\mathbb{N}} u_j^2)$ (by C-S)

$\leq \sum_{i,j\in\mathbb{N}\times\mathbb{N}} (i^s K_{ij})^2 \sum_{j\in\mathbb{N}} u_j^2 = K^+ \|u\|_{\ell^2}^2$

- $v = Ku$ $\therefore \sup_{u\in H} \|Ku\|_{\ell^2} / \|u\|_{\ell^2} \leq \sup_{u\in H} \|Ku\|_{\mathcal{H}^s} / \|u\|_{\ell^2} \leq K^+ < \infty$ $\therefore K \in \mathcal{L}(H,H)$
- If $\|u\|_{\ell^2}^2 < R^2$ then $\|Ku\|_{\mathcal{H}^s}^2 < K^+ R^2$. Thus K is a compact operator on H

$\therefore K = \sum_{j=1}^\infty \lambda^{(j)} w^{(j)} \otimes w^{(j)}$.

$$\text{(SVD)} \left\{ \begin{array}{l} Tv^{(j)} = \sigma^{(j)} u^{(j)} \\ \|v^{(j)}\| = \|u^{(j)}\| = 1 \end{array} \right\}$$

<u>Proposition:</u> If $T \in B_{00}$ has rank r then $\{u^{(j)}\}_{j=1}^r$, $\{v^{(j)}\}_{j=1}^r$ & $\{\sigma^{(j)}\}_{j=1}^r$ solving (SVD) & with $\langle u^{(j)}, u^{(k)}\rangle = \delta_{jk}$ $\langle v^{(j)}, v^{(k)}\rangle = \delta_{jk}$, $\sigma^{(j)} \in \mathbb{R}^+$: $T = \sum_{j=1}^r \sigma^{(j)} u^{(j)} \otimes v^{(j)}$

<u>Theorem:</u> If $T \in B_0 \setminus B_{00}$ then $\exists \{u^{(j)}\}_{j=1}^r$, $\{v^{(j)}\}_{j=1}^r$ orthonormal sets, $\{\sigma^{(j)}\}_{j=1}^\infty \in \mathbb{R}^+$, $T = \lim_{r\to\infty} T_r$ in B where $T_r = \sum_{j=1}^r \sigma^{(j)} u^{(j)} \otimes v^{(j)}$

<u>Example:</u> $(Lu)_j = u_{j+1} - \lambda u_j$, $j \in \mathbb{N}$. Identified L^*. $|\lambda| < 1$.

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & L \end{pmatrix}}_{A} u = \begin{pmatrix} u_1 \\ Lu \end{pmatrix}, \quad B = \text{diag}\{\lambda^j\}, \quad (Bh)_j = \lambda^j h_j$$

$Au = Bv \qquad u_1 = \lambda v_1$

$u_{j+1} - \lambda u_j = \lambda^{j+1} v_{j+1}$, $j \in \mathbb{N}$.

$u_j = \lambda^j z_j$; $z_{j+1} - z_j = w_{j+1}$, $z_1 = w_1 \Rightarrow z_j = \sum_{\ell=1}^j w_\ell$

$\Rightarrow \forall j \ |z_j|^2 \leq j \sum_{\ell=1}^j w_\ell^2 \leq j \|w\|_{\ell^2}^2$.

Fix any $s > 0$. $\|u\|_{\mathcal{H}^s}^2 = \sum_{j\in\mathbb{N}} j^{2s} u_j^2 = \sum_{j\in\mathbb{N}} j^{2s} \lambda^{2j} z_j^2$

$\leq (\sum_{j\in\mathbb{N}} j^{2s} \lambda^{2j} j) \|w\|_{\ell^2}^2 \rightsquigarrow \exists C^+ > 0: \ \leq C^+ \|w\|_{\ell^2}^2$

"A^{-1} exists"

$\|A^{-1}B\|_{\mathcal{L}(H,H)}^2 = \sup_{w\in\ell^2} \|u\|_{\ell^2}^2 / \|w\|_{\ell^2}^2 \leq \sup_{w\in\ell^2} \|u\|_{\mathcal{H}^s}^2 / \|w\|_{\ell^2}^2 \leq C^+$

Also compact because $\|A^{-1}Bw\|_{\mathcal{H}^s}^2 \leq C^+ \|w\|_{\ell^2}^2$

$A^{-1}B$ is compact $\ell^2 \to \ell^2$

11/21 Cont'd:

Theorem: If $Re(\lambda) < 0$ $\forall$ eigenvalues of A then
$\exists\, C \in (0,\infty)$: $\sup_{t\geq 0} \|e^{At}\| = C$.

~~Lemma~~ Calculus:

$Re(\lambda) < 0$, $m \geq 0$ $\quad \sup_{t\geq 0} |t^m e^{\lambda t}| < \infty$

$Re(\lambda) > 0$ $\quad m \geq 0$, $\delta > 0$ $\quad \sup_{t\geq 0} \dfrac{|t^m e^{\lambda t}|}{|e^{(\lambda+\delta)t}|} < \infty$

If $\exists$ e.v. λ s.t. $Re(\lambda) = \alpha > 0$ then
$\exists\, C \in (0,\infty)$, $\delta > 0$ $\quad \sup_{t\geq 0} \dfrac{\|e^{At}\|}{e^{(\alpha+\delta)t}} = C$

~~Theorem~~ Remark: C is not necessarily attained at x_0 due to transience!

Theorem: Let $\rho(A) < 1$. Then $\exists\, \alpha \in (\rho(A), 1)$ & norm $\|\cdot\|$ on $\mathbb{C}^n$ such that in the induced matrix norm $\|A^k\| \leq \alpha^k$.

Lemma: Let $\delta > 0$. Then $\exists\, \|\cdot\|$ on $\mathbb{C}^n$ such that, in the induced norm
$$\rho(A) \leq \|A\| \leq \rho(A) + \delta.$$
(equality if A is symmetric)

proof: let $A\varphi = \lambda\varphi$, $\rho(A) = |\lambda|$
$$\|A\| = \sup_{u\neq 0} \frac{\|Au\|}{\|u\|} \geq \frac{\|A\varphi\|}{\|\varphi\|} = \frac{\|\lambda\varphi\|}{\|\varphi\|} = |\lambda| = \rho(A)$$
$$J_k^\delta(\lambda) = \begin{pmatrix} \lambda & \delta & & 0 \\ & \ddots & \ddots & \\ & & & \delta \\ 0 & & & \lambda \end{pmatrix} \in \mathbb{C}^{k\times k} \qquad J_k(\lambda) = J_k^{\delta=1}(\lambda)$$
$D_k^\delta = \mathrm{diag}\{\delta, \delta^2, \ldots, \delta^k\} \in \mathbb{C}^{k\times k}$. Then
$$(D_k^\delta)^{-1} J_k(\lambda) D_k^\delta = J_k^\delta(\lambda)$$
$$D^\delta = \mathrm{diag}\{D_{n_1}^\delta, D_{n_2}^\delta, \ldots, D_{n_k}^\delta\}$$
$$(D^\delta)^{-1} J D^\delta = J^\delta$$
$$J^\delta = \mathrm{diag}\{J_{n_1}^\delta(\lambda), J_{n_2}^\delta(\lambda), \ldots, J_{n_k}^\delta(\lambda)\}$$
$$A = SJS^{-1} = SD^\delta J^\delta (SD^\delta)^{-1}$$
$$\|u\|_{\mathbb{C}^n} := \|(SD^\delta)^{-1} u\|_\infty$$
$$\|A\|_{\mathbb{C}^n} = \sup_{u\neq 0} \frac{\|(SD^\delta)^{-1} A u\|_\infty}{\|(SD^\delta)^{-1} u\|_\infty} = \sup_{u\neq 0} \frac{\|J^\delta (SD^\delta)^{-1} u\|_\infty}{\|(SD^\delta)^{-1} u\|_\infty}$$
$$= \sup_{v\neq 0} \frac{\|J^\delta v\|_\infty}{\|v\|_\infty} = \|J^\delta\|_\infty = \max_{e.v.} |\lambda| + \delta = \rho(A) + \delta$$

$\rightarrow$ proof: $\delta = \dfrac{1-\rho(A)}{2}$ $\quad \rho(A) + \delta = \dfrac{1+\rho(A)}{2} < 1$

induced $\|A^k\| \leq \|A\|^k \leq \left(\dfrac{1+\rho(A)}{2}\right)^k$

11/26 Fixed point Theorems:

$f(x) = 0$ $\quad f: \mathbb{R}^n \to \mathbb{R}$

$g(x) := f(x) + x$; $g(x) = x \iff f(x) = 0$

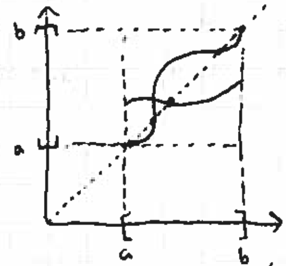
$(X, \|\cdot\|)$ Banach over $\mathbb{R}$; $T: X \to X$ (possibly nonlinear)

$T' = T + I$

We're looking at $Tx = x$

Theorem: [Brouwer/Schauder]
Every continuous function from a convex compact subset $K \subseteq X$ to itself, $T: K \to K$ has a fixed point

Example: $\mathbb{R}$ $\quad f: [a,b] \to [a,b]$ is continuous
f has a fixed point : $\exists\, x^* \in \mathbb{R}$: $f(x^*) = x^*$



[2] Contraction Mapping/Banach Fixed Point Theorem
$(X, \|\cdot\|)$ Banach over $\mathbb{R}$; $T: X \to X$

Definition: Let M be a closed non-empty subset in X. Then $T: X \to X$ is a contraction on M (is contractive on M) if:
1) $T(M) \subseteq M$
2) $\exists\, \lambda \in (0,1)$ s.t. $\sup_{u\neq v,\ u,v\in M} \dfrac{\|T(u) - T(v)\|}{\|u - v\|} \leq \lambda$

$\equiv \|T(u) - T(v)\| \leq \lambda \|u - v\|$ $\quad \forall\, u, v \in M$

Remark: If T is linear and M is a subspace then
T contraction $\iff \|T\|_{\mathcal{L}(M,M)} \leq \lambda$ $\quad \lambda \in (0,1)$

Example: $f(x) = \cos x$ on $\mathbb{R}$
$f(x)$ is not a contraction as $f: \mathbb{R} \to \mathbb{R}$
By contradiction, $|\cos x - \cos y| \leq \lambda |x - y|$ $\quad \forall\, x, y \in \mathbb{R}$
$\Rightarrow \dfrac{|\cos x - \cos y|}{|x-y|} \leq \lambda \xrightarrow[y\to x]{\lim} |\sin x| \leq \lambda \rightsquigarrow$ contradiction b/c $\lambda \in (0,1)$

$f_r : [0,1] \to [0,1] = f|_{[0,1]}$
f_r is a contraction:
(1) $\cos([0,1]) = [\cos 1, 1] \subset [0,1]$
(2) Use Mean value theorem : $\dfrac{f(x) - f(y)}{x - y} = f_r'(t)$ $\quad t \in (x,y)$
$\dfrac{\cos x - \cos y}{x - y} = \sin(t) \leq \sin(1) < 0.9.$

ACM/CMS 107

11/5 Recitation:

- Sample a random function $f:[0,2\pi]\to\mathbb{R}$
  - $\to f=\sum_{j=1}^{\infty} a_n \sin(jx)+b_n\cos(jx)$
  - $\to \{a_n\},\{b_n\}$ are random

Hilbert-Schmidt:

- Let $D=[a,b]\subset\mathbb{R}$ and $\mathcal{K}:D\times D\to\mathbb{R}$
- $\mathcal{K}$ is called a Hilbert-Schmidt kernel if $\mathcal{K}\in L^2(D\times D)$
- Define operator $\mathcal{K}$ on $L^2(D)$ by
  $(\mathcal{K}u)(x)=\int_D \mathcal{K}(x,y)u(y)\,dy \quad \forall u\in L^2(D)$
- $\mathcal{K}$ is a Hilbert-Schmidt operator.
- Since D is compact, $\mathcal{K}\in C(D\times D)$ is Hilbert-Schmidt.

Theorem: the following hold:

(i) $\mathcal{K}:L^2(D)\to L^2(D)$
(ii) $\mathcal{K}$ is linear
(iii) $\mathcal{K}$ is bounded
(iv) $\mathcal{K}$ is compact (hard)
(v) $\mathcal{K}$ is self-adjoint (symmetric) iff $\mathcal{K}(x,y)=\mathcal{K}(y,x)$

Lemma: For any Banach space X, the set of compact operators on X are closed linear subspaces of $\mathcal{L}(X)$

Theorem (Mercer): Suppose $\mathcal{K}\in C(D\times D)$ and symmetric and that the associated Hilbert Schmidt operator $\mathcal{K}$ is non-negative ($\langle u,\mathcal{K}u\rangle\geq 0$). Let $\{\lambda_j\},\{\varphi_j\}$ be the eigendecomposition of $\mathcal{K}$. Then
$$\mathcal{K}(x,y)=\sum_{j=1}^{\infty}\lambda_j\varphi_j(x)\varphi_j(y).$$

Stochastic Processes:

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space. Then define
$L^2(\Omega)=\{X:\Omega\to\mathbb{R}:\mathbb{E}[|X|^2]<\infty\}$ and
$\langle X,Y\rangle=\mathbb{E}[XY]$

- Define L^2-stochastic processes $X:D\times\Omega\to\mathbb{R}$ s.t.
  $X(t;\cdot):=X_t\in L^2(\Omega)\quad\forall t\in D$
- We say a s.p. X is centered if $\mathbb{E}[X_t]=0\ \forall t$
- A s.p. is mean-squared continuous if
  $\lim_{\varepsilon\to 0}\mathbb{E}[(X_{t+\varepsilon}-X_t)^2]=0\quad\forall t$
- Define autocorrelation function of X
  $R_X:D\times D\to\mathbb{R}$ by $R_X(s,t)=\mathbb{E}(X_sX_t)$

Lemma: A L^2-s.p. X is mean-squared continuous iff R_X is continuous.

11/21

$A^T=A\Rightarrow Q\Lambda Q^T$

$\Lambda=\mathrm{diag}\{\lambda_1,\dots,\lambda_n\}\quad \rho(A)=\max_{1\le j\le n}|\lambda_j|$

$\|e^{At}\|=e^{\rho(A)t}\ ;\ \|A^k\|=(\rho(A))^k$

$f(A)$// $f:\mathbb{R}^{n\times n}\circlearrowleft$. $f:\mathbb{C}\circlearrowleft\ f(z)=\sum_{j=0}^{\infty}a_jz^j$

$f(A)=\sum_{j=0}^{\infty}a_jA^j$ radius of convergence $r>0$.

JNF// $J_k(\lambda)=\begin{pmatrix}\lambda & 1 & & \\ & \ddots & \ddots & \\ & & & 1\\ & & & \lambda\end{pmatrix}\in\mathbb{C}^{k\times k}\quad\sum_{l=1}^{k}n_l=n$

$J=\mathrm{diag}\{J_{n_1}(\lambda_1),\dots,J_{n_k}(\lambda_k)\}\in\mathbb{C}^{n\times n}$

$A=SJS^{-1}\to A^j=SJ^jS^{-1}$

$J^j=\mathrm{diag}\{J_{n_1}^j(\lambda_1),\dots,J_{n_k}^j(\lambda_k)\}$ (*)

Combining// $f(A)=Sf(J)S^{-1}$.

By (*) knowing $f(J_k(\lambda))$ enables us to define $f(A)$ for any A.

Ex. If $A^T=A$ then $S=Q$, $J=\Lambda$, $J_{n_l}(\lambda_l)=\lambda_l$.
$\{\lambda_l\}_{l=1}^n$ eigenvalues of A.
$\Lambda=\mathrm{diag}\{\lambda_1,\dots,\lambda_n\}$.
$e^{Jt}=\mathrm{diag}\{e^{\lambda_1t},\dots,e^{\lambda_nt}\}$
$e^{At}=Qe^{Jt}Q^T$.

e^{At}// $A\neq A^T$ in general.

$f(z)=e^{zt}$

$\dot u=J_k(\lambda)u,\ u(0)=v\ ;\ u(t)=e^{J_k(\lambda)t}v$

$\dot u_1=\lambda u_1(t)+u_2(t),\ u_1(0)=v_1$

$\dot u_2=\lambda u_2(t)+u_3(t),\ u_2(0)=v_2$

$\vdots$

$\dot u_{k-1}=\lambda u_{k-1}(t)+u_k(t),\ u_{k-1}(0)=v_{k-1}$

$\dot u_k=\lambda u_k(t),\quad u_k(0)=v_k$

Note: $f(J)=\mathrm{diag}\{f(J_{n_1}(\lambda_1)),\dots,f(J_{n_k}(\lambda_k))\}$

Lemma: $u_l(t)=\sum_{j=l}^{k}\frac{t^{(j-l)}}{(j-l)!}e^{t\lambda}v_j$

Proof: $u_l(0)=v_l$

$\dot u_l(t)=\sum_{j=l+1}^{k}\frac{t^{j-l-1}}{(j-l-1)!}e^{t\lambda}v_j+\lambda u_l(t)$

$=\begin{cases}u_{l+1}(t)+\lambda u_l(t) & l<k\\ \lambda u_l(t) & l=k\end{cases}$

Theorem: $f(J_k(\lambda))=\begin{pmatrix}f(\lambda) & f'(\lambda) & \frac{1}{2!}f''(\lambda) & \cdots & \frac{1}{(k-1)!}f^{(k-1)}(\lambda)\\ & f(\lambda) & \ddots & & \vdots\\ & & \ddots & \ddots & \\ 0 & & & & \end{pmatrix}$

Remark: JNF $\sum_{l=1}^{k}n_l=n$.

To define $f(A)$ need $f\in C^p$ where $p=\max_l n_l-1$

11/26 cont'd:

Theorem [Banach fixed-point]:

Let T be a contractive on M. Then the equation $u = T(u)$ (*)

(1) • has a solution $u \in M$

(4) • this solution is unique in M

(2) • iteration $u_{n+1} = T(u_n)$ with $u_0 \in M$ converges to $u \in M$ solving (*)

(3) • $\|u_n - u\| \le \frac{\lambda^n}{1-\lambda}\|u_1 - u_0\| \quad \forall n \in \mathbb{N}$

Proof: Fix any $u_0 \in M$; Show that $\{u_n\}_{n=1}^{\infty}$ is Cauchy in X. By induction, every $u_n \in M$.

$$\|u_{n+1} - u_n\| = \|T(u_n) - T(u_{n-1})\| \overset{\text{contractive}}{\le} \lambda\|u_n - u_{n-1}\| \le \dots \le \lambda^n \|u_1 - u_0\|$$

$$\|u_{n+m} - u_n\| \le \|u_{n+m} - u_{n+m-1}\| + \dots + \|u_{n+1} - u_n\|$$
$$\le (\lambda^{n+m-1} + \dots + \lambda^n)\|u_1 - u_0\| \le \lambda^n \sum_{k=0}^{\infty} \lambda^k \|u_1 - u_0\|$$
$$= \lambda^n (1-\lambda)^{-1}\|u_1 - u_0\|. \Rightarrow \{u_n\} \text{ is Cauchy.}$$

$u \in M$ because M is closed and $u_n \in M$

Thus $T(u) \in M$. Moreover,

$$\|T(u_n) - T(u)\| \le \lambda \|u_n - u\|$$

(because T is contractive)

Therefore $T(u_n) \to T(u)$

$$\|u - T(u)\| \le \|u - u_{n+1}\| + \|u_{n+1} - T(u)\| =$$
$$= \|u - u_{n+1}\| + \|T(u_n) - T(u)\|, \text{ let } n \to \infty$$

(1), (2), (3) ✓, Now (4)!

For contradiction, assume $u = T(u)$, $v = T(v)$; $u, v \in M$

$$\|u - v\| = \|T(u) - T(v)\| \le \lambda \|u - v\|$$

$\|u - v\| \ne 0 \Rightarrow 1 \le \lambda \Rightarrow$ contradiction.

Applications to ODEs

Definition: $B(v, r) = B_{\mathbb{R}^n}(v, r) \rightsquigarrow$ Ball centered at v of radius r

Definition: $f: \mathbb{R}^n \to \mathbb{R}$ is Lipschitz if $\exists L \in (0, \infty)$ s.t.

$$\sup_{u \ne v;\, u, v \in \mathbb{R}^n} \frac{\|f(u) - f(v)\|}{\|u - v\|} \le L$$

Definition: $f: \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz if $\forall R > 0$ $\exists L = L(R) \in (0, \infty)$ s.t. $\sup_{u \ne v;\, u, v \in \mathbb{R}^n} \frac{\|f(u) - f(v)\|}{\|u - v\|} \le L(R)$

$du/dt = f(u)$, $u(0) = u_0$ (ODE)

$u(t) = u_0 + \int_0^t f(u(s))\,ds$ (IE)

u solves (ODE) $\Rightarrow$ u solves (IE)

u solves (IE) $+ u \in C^1 \Rightarrow u$ solves (ODE)

$X = C([0,T]; \mathbb{R}^n)$; $\|u\|_X = \sup_{t \in [0,T]} \|u(t)\|_{\mathbb{R}^n}$

$u_0(t) := u_0 \quad \forall t \in [0, T]$

Fix $r > 0$, define $M := B(u_0, r)$

Assume f is locally Lipschitz with $L(R)$ on $B(0, R)$.

Choose $R^* > r + \|u_0\|_{\mathbb{R}^n} \Rightarrow M \subseteq B(0, R^*)$

Choose $T^+ > 0$: $\begin{cases} T^+ L(R^*) \le 1/2 \\ T^+ \|f(u_0)\|_{\mathbb{R}^n} \le r/2. \end{cases}$

Theorem: (IE) has a unique solution in M

Uniform Contraction Principle:

$(X, \|\cdot\|)$ Banach

$(\Theta, \|\cdot\|_\Theta)$ Banach $\rightarrow$ intuitively, space of parameters.

$T: X \times \Theta \to X$ "$T(u, \theta) = u$"

We want: $\exists! u(\theta) \in M \quad \forall \theta \in D$

$u: \Theta \to X$ [We want $u(\cdot)$ cts.]

Apply Banach fixed point theorem: $\exists!\, u(\theta)\ \forall \theta \in D$

$$u(\theta) = T(u(\theta), \theta)$$

(i) $T(\cdot, \theta): X \to X$ is contractive on M $\forall \theta \in D$ with $\lambda(\theta) \in (0, 1)$

Fix $\theta \in D$. $\|u(\theta) - u(\varphi)\| < \varepsilon \quad \forall \varepsilon > 0$ provided $\|\theta - \varphi\|_\Theta < \delta$

$$\|u(\theta) - u(\varphi)\| = \|T(u(\theta), \theta) - T(u(\varphi), \varphi)\| \le \|T(u(\theta), \theta) - T(u(\theta), \varphi)\|$$
$$+ \|T(u(\theta), \varphi) - T(u(\varphi), \varphi)\|$$

$$\le \lambda(\varphi)\|u(\theta) - u(\varphi)\|$$

$$\le \|T(u(\theta), \theta) - T(u(\theta), \varphi)\| + \alpha\|u(\theta) - u(\varphi)\|$$

(ii) $\sup \lambda(\theta) \le \alpha < 1$

(iii) $T(x_j): \Theta \to X$ is continuous

$$\le (1 - \alpha)\varepsilon + \alpha\|u(\theta) - u(\varphi)\|$$

∎

L18: Gradient Descent

Based on: → see assumption 2.

$\frac{du}{dt} = -K\nabla\Phi(u)$ (ODE) $u_0 = u(0)$

$\frac{d}{dt}\{\Phi(u(t))\} = \sum_{j=1}^{J} \partial\Phi/\partial u_j(u(t)) \, du_j/dt(t)$

$= \langle \nabla\Phi(u(t)), du/dt\rangle = -|K^{1/2} du/dt|^2 \le 0$

ODE: $\langle\cdot,\cdot\rangle$, $|\cdot|$ Euclidean

Assumption 1:

- $\Phi \in C^2(\mathbb{R}^n;\mathbb{R}^+)$
- For every $R>0$ $\exists\, r>0$: $\Phi(u)\le R \Rightarrow |u|<r$

Theorem: Under assumption(s) 1 (& 2) has a unique solution $u\in C^1([0,\infty),\mathbb{R}^n)$ for every $u_0\in\mathbb{R}^n$. Furthermore, $\Phi(u(t))\le\Phi(u_0)$ $\forall$ $t\ge 0$.

Proof: $\nabla\Phi\in C^1(\mathbb{R}^n;\mathbb{R}^n) \Rightarrow$ (ODE theory)

$\exists\, T^*(u_0)$: $\exists!$ solution $u(t)$, $t\in[0,T^*(u_0))$.

Furthermore, if the maximal $T^*(u_0)$ is $<\infty$ then (ODE theory) states that $\lim_{t\uparrow T^*(u_0)} |u(t)| = \infty$. But, for all $t\in[0,T^*(u_0))$

$\Phi(u(t)) \le \Phi(u(0)) = \Phi(u_0)$ and thus $\exists$ $r<\infty$: $\sup_{t\in[0,T^*(u_0))} |u(t)|<r$. $\therefore T^*(u_0)=\infty$

Remark: If $\exists$ $t_j\to\infty$ & u^*: $u(t_j)\to u^*$ as $t_j\to\infty$ then $\nabla\Phi(u^*)=0$.

Assumption 2:

$K\in\mathbb{R}^{n\times n}$ is symmetric, positive-definite

Ex: Heat Equation 1: $\langle\cdot,\cdot\rangle$, $\|\cdot\|$ on $L^2(I;\mathbb{R})$, $I=(0,1)$

$\partial_t u = \partial_x^2 u$, $(x,t)\in(0,1)\times(0,\infty)$

$u=0$ $\quad (x,t)\in\{0,1\}\times(0,\infty)$

$u=u_0$ $\quad (x,t)\in(0,1)\times\{0\}$

$\Phi_0(u) = \frac{1}{2}\int_0^1 u(x)^2 dx = \frac{1}{2}\|u\|_{L^2}^2$

$\Phi_0: L^2\to\mathbb{R}^+$

$K = -\frac{d^2}{dx^2}$ with Dirichlet boundary conditions at $\{0,1\}$

$k: D(k)\to L^2$

$D(k) = H^2(I;\mathbb{R})\cap H_0^1(I;\mathbb{R})$

$\partial_t u = -k\, D\Phi_0(u)$

$\Phi_0(u+h) = \frac{1}{2}\int_0^1 (u(x)+h(x))^2 dx = \Phi_0(u) + \langle u,h\rangle + \frac{1}{2}\|h\|_{L^2}^2$

$= \Phi_0(u) + \langle D\Phi_0(u),h\rangle + \frac{1}{2}\|h\|_{L^2}^2$

↳ $D\Phi(u)$ is just u, justifying

$K^{-1}\partial_t u = -D\Phi_0(u)$

$-\langle\partial_t u, k^{-1}\partial_t u\rangle = \langle D\Phi_0(u),\partial_t u\rangle$

$-\|K^{-1/2}\partial_t u\|_{L^2}^2 = \frac{d}{dt}\Phi_0(u(t))$

$\frac{d}{dt}\int_0^1 \frac{1}{2}u(x,t)^2 dt = \int_0^1 u(x,t)\,\partial_t u(x,t)\,dx$

$= \int_0^1 u(x,t)\,\partial_x^2 u(x,t)\,dx$

$= -\int_0^1 |\partial_x u(x,t)|^2 dx = -\|\partial_x u\|_{L^2}^2$

$(= -\|K^{-1/2}\partial_t u\|_{L^2}^2)$

Heat Equation 2:

$\frac{d}{dt}\int_0^1 \frac{1}{2}|\partial_x u(x,t)|^2 dx = \int_0^1 \partial_x u(x,t)\,\partial_{xt}^2 u(x,t)\,dx$

$= -\int_0^1 \partial_x^2 u(x,t)\,\partial_t u(x,t)\,dx$

$= -\int_0^1 |\partial_t u(x,t)|^2 dx = -\|\partial_t u\|_{L^2}^2$.

$\Phi_1(u) = \frac{1}{2}\int_0^1 \left|\frac{du}{dx}(x)\right|^2 dx$

$\Phi_1: H_0^1 \to \mathbb{R}^+$

$\Phi_1(u+h) = \Phi_1(u) + \int_0^1 \frac{du}{dx}\frac{dh}{dx}dx + \frac{1}{2}\left\|\frac{dh}{dx}\right\|_{L^2}^2$.

$= \Phi_1(u) - \int_0^1 \frac{d^2u}{dx^2}h\,dx + \frac{1}{2}\left\|\frac{dh}{dx}\right\|_{L^2}^2$

$= \Phi_1(u) + \langle -\frac{d^2u}{dx^2}, h\rangle + \frac{1}{2}\left\|\frac{dh}{dx}\right\|_{L^2}^2$

$D\Phi_1(u) = -\frac{d^2u}{dx^2}$

Heat Equation 3:

$F:\mathbb{R}\to\mathbb{R}$

$\Phi_F(u) = \int_0^1 F(u(x))dx$

$\Phi_F(u+h) = \int_0^1 F(u(x)+h(x))dx$

$= \Phi_F(u) + \underbrace{\int_0^1 F'(u(x))h(x)dx}_{D\Phi_F(u(x))} + o(\|h\|_{L^2}^2)$

$\Phi_2(u) = \int_0^1 u(x)\ln(u(x))dx$

$D\Phi_2(u)(x) = 1 + \ln u(x)$

$\partial_t u = \partial_x(\partial_x u) = \partial_x(u\,\partial_x[\ln u + 1])$

$\partial_t u = \partial_x(u)\partial_x\, D\Phi_2(u(x))$.

$\partial_t u = -K(u)\,D\Phi_2(u)$.

$K(u)^{-1}\partial_t u = -D\Phi_2(u)$

$-\langle\partial_t u, k(u)^{-1}\partial_t u\rangle = d\Phi_2/dt(u(t))$.

HE 1	Gradient	H^{-1}	$(K^{-1/2})$	$\int_0^1 u^2 dx$
HE 2	Gradient	L^2	$(K\equiv I)$	$\int_0^1 u\ln u\,dx$
HE 3	Gradient	Wasserstein	$(K(u))$	$\int_0^1 |\partial_x u|^2 dx$