10/2 L1

Linear Spaces & operators:

$x \in \mathbb{R}^n$ — linear space

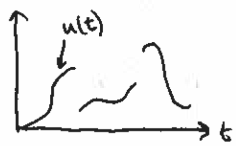
$\forall\, x_1, x_2 \in \mathbb{R}^n$
$\forall\, \alpha_1, \alpha_2 \in \mathbb{R}$ $\quad \alpha_1 x_1 + \alpha_2 x_2 \in \mathbb{R}^n$

Linear space V

- V is a set
- Addition: $v_1, v_2 \in V \to v_1 + v_2 \in V$
- Scaling over $\mathbb{F}$: $v \in V$, $\alpha \in \mathbb{F}$ $\alpha v \in V$
- $0 \in V$ s.t. $v + 0 = v$
- $\mathbb{1} \in \mathbb{F}$ s.t. $\mathbb{1} v = v$

Ex: $V = P[0,T]$ ~ signals, piecewise cnts. on $[0,T]$ — finite # of discontinuities — intervals are finite duration

$u(\cdot) \in P[0,T]$ over $\mathbb{R}$ field



$u_1(\cdot) + u_2(\cdot) = (u_1 + u_2)(\cdot)$

$u : T \to V$

↳ ∞ dimensional because it's continuous on $[0,T]$

zero element: $0(\cdot) \rightsquigarrow 0(t) = 0 \in \mathbb{R}$

Ex: $V = P[0,\infty)$

Ex: $V = C^m[0,T]$ m-differentiable cnts. function

e^{at} is C^∞

Norms of signals:

$\|\cdot\|$ $\quad u(\cdot) \in \mathcal{V}$ $\quad \|\cdot\| : \mathcal{V} \mapsto \mathbb{R}$

satisfies: a) $\|u(\cdot)\| \geq 0 \quad \forall\, u(\cdot) \in \mathcal{V}$

b) $\|u(\cdot)\| = 0$ iff $u(\cdot) = 0$

c) $\|u(\cdot) + v(\cdot)\| \leq \|u(\cdot)\| + \|v(\cdot)\|$

d) $\|\alpha u(\cdot)\| = |\alpha| \cdot \|u(\cdot)\|$

Ex: $v \in \mathbb{R}^n$

$\|v\|_2 = \sqrt{v_1^2 + \dots + v_n^2}$

$u(\cdot) \in \mathcal{V}$

$\|u(\cdot)\|_2 = \left(\int_0^T \|u(\tau)\|^2 d\tau\right)^{1/2}$

★ Try to show c) on ↑

Extends to p-norm:

$\|u(\cdot)\|_p = \left(\int_0^T \|u(\tau)\|^p d\tau\right)^{1/p}$

$\|u(\cdot)\|_\infty = \max_{\tau \in [0,T]} \|u(\tau)\|$

$= \sup_{t \in \mathcal{T}} \|u(t)\|$

Ex: $\mathcal{T} = [0,\infty)$ $\quad u(t) = 1 - e^{-t}$, sup $= 1$

Mappings:

$$u(\cdot) \xrightarrow{} \boxed{L} \xrightarrow{} y(\cdot), \quad u \in \mathcal{U}, \; y \in \mathcal{Y}$$

$y(\cdot) = L(u(\cdot)) \quad L : \mathcal{U} \to \mathcal{Y}$

L is a linear operator if:

a) $L(u_1 + u_2) = L(u_1) + L(u_2)$

b) $L(\alpha u_1) = \alpha L(u_1)$

Ex: • Time delay (not finite dimensional, but still linear)

• $Av = \int_0^1 v(t)\,dt$, $v \in C[0,1]$

$A : C[0,1] \to \mathbb{R}$

Given: $h \in C[0,\infty)$, $v \in [0,\infty)$

$(C_h v)(t) = \int_0^t h(t-\tau) v(\tau)\, d\tau$ $\Big\}$ impulse response

$C_h : C[0,\infty) \to C[0,\infty)$

Dynamical System: $D = (\mathcal{U}, \Sigma, \mathcal{Y}, s, r)$ — r: read out map; s: state update map

on $\mathcal{T} \subset \mathbb{R}$ (time interval)

$\mathcal{U}$: input space of signals; Σ: "state space" (generally $\mathbb{R}^n$); $\mathcal{Y}$: output space of signals

x is Σ in
$\dot{x} = Ax + Bu$
$y = Cx$

Update map: $s(t_1, t_0, x_0, u(\cdot))$ = state at time t_1 given that state at $t_0 = x_0$ & apply $u(t)$, $t \in [t_0, t_1]$

$s \in \Sigma : \mathcal{T} \times \mathcal{T} \times \Sigma \times \mathcal{U} \to \Sigma$

a) State transition axiom: If two signals $u(\cdot)$ & $\tilde{u}(\cdot)$ agree on $\mathcal{T}$, then $s(t_1, t_0, x_0, u(\cdot)) = s(t_1, t_0, x_0, \tilde{u}(\cdot))$ $\forall\, t \in [t_0, t_1]$

↳ $u(t) = \tilde{u}(t) \quad \forall\, t \in \mathcal{T} = [t_0, t_1]$

b) Semi-group:

$s(t_2, t_1, s(t_1, t_0, x_0, u(\cdot)), u(\cdot)) = s(t_2, t_0, x_0, u(\cdot))$

10/4 L2: Impulse Response:

- zero state response ($x_0 = 0$) to $u(t)$

$G(t): y(t) = \int_{-\infty}^{\infty} G(t-\tau) u(\tau)\, d\tau := G * u$

- Laplace transform: $y(t) \to \hat{Y}(s)$

$\hat{Y}(s) = \int_0^\infty y(t) e^{-st}\, dt$

Now we will plug in the zero state response,

$$\hat{Y}(s) = \int_0^\infty \left(\int_0^\infty G(t-\tau) u(\tau)\, d\tau\right) e^{-st}\, dt$$
$$= \int_0^\infty \left(\int_0^\infty G(t-\tau) e^{-st}\, dt\right) u(\tau)\, d\tau$$
$$= \int_0^\infty \left(\int_0^\infty G(t-\tau) e^{-s(t-\tau)}\, d(t-\tau)\right) u(\tau) e^{-s\tau}\, d\tau \quad \therefore \tilde{t} = t - \tau$$
$$= \int_0^\infty \left(\int_0^\infty G(\tilde{t}) e^{-s\tilde{t}}\, d\tilde{t}\right) u(\tau) e^{-s\tau}\, d\tau$$
$$= \underbrace{\left(\int_0^\infty G(\tilde{t}) e^{-s\tilde{t}}\, d\tilde{t}\right)}_{\hat{G}(s)} \underbrace{\left(\int_0^\infty u(\tau) e^{-s\tau}\, d\tau\right)}_{\hat{U}(s)}$$

$\hat{Y}(s) = \hat{G}(s)\hat{U}(s) = \hat{G}_{yu}(s)\hat{U}(s)$

Norms of system (SISO)

$\|\hat{G}\|_2 = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{G}(j\omega)|^2 d\omega\right)^{1/2}$

$\|\hat{G}\|_\infty = \sup_\omega |\hat{G}(j\omega)|$

Parseval's theorem:

$\|\hat{G}(s)\|_2^2 = \int_{-\infty}^{\infty} |G(t)|^2 dt = \|G(t)\|_2^2$

(∞-norm)

$\|\hat{G}\hat{H}\|_\infty \le \|\hat{G}\|_\infty \|\hat{H}\|_\infty$

State space:

$\dot{x} = Ax + Bu \;;\; y = Cx + Du$

$\hat{G}(s) = C(sI - A)^{-1}B + D$ (from $\mathcal{L}$)

$\hat{G}(s) = n(s)/d(s)$; pole = roots of $d(s)$; zero = " $n(s)$

$\hat{G}(s)$ is proper if $\lim_{\omega\to\infty} \hat{G}(j\omega)$ is finite
$\Leftrightarrow \deg(d) \ge \deg(n)$

strictly proper $\hat{G}(j\infty) = 0$
$\Leftrightarrow \deg(d) > \deg(n)$

biproper $\hat{G}(s)$ & $\hat{G}^{-1}(s)$ are proper
$\Leftrightarrow \deg(d) = \deg(n)$

stable if analytic in the closed RHP
$\Leftrightarrow$ No RHP poles

Theorem: 2-norm of rational $\hat{G}(s)$ is finite iff $\hat{G}$ is strictly proper, with no poles on Im axis.
... ∞-norm — " — proper, with — " —

Induced Norm of system: (system gain)

$\|G\| = \sup_{\|u\|_a \le 1} \|y\|_b$ where $y = G * u$

	$\|u_2\|$	$\|u_\infty\|$
$\|y\|_2$	$\|\hat{G}\|_\infty$ ①	∞
$\|y\|_\infty$	$\|\hat{G}\|_2$	$\|G\|_1$

Need to show that "=" = "$\ge$" + "$\le$"

$\hat{G}$ stable & strictly proper

① $\sup_{\|u\|_2 \le 1} \|y\|_2 = \|\hat{G}\|_\infty$

proof: "$\le$" $\|y\|_2^2 = \|\hat{Y}(s)\|_2^2 = \|\hat{G}(s)\hat{U}(s)\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{G}(j\omega)|^2 |\hat{U}(j\omega)|^2 d\omega$

$\le \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\sup_\omega |\hat{G}(j\omega)|^2}_{\|\hat{G}\|_\infty^2} \underbrace{|\hat{U}(j\omega)|^2 d\omega}_{\|\hat{U}\|_2^2} = \|\hat{G}\|_\infty^2 \|\hat{U}\|_2^2 = \|\hat{G}\|_\infty^2 \|u\|_2^2$

... $\|y\|_2^2 \le \|\hat{G}\|_\infty^2 \|u\|_2^2$

"$\ge$": Find a "u" s.t. $\|y\|_2 / \|u\|_2 = \|\hat{G}\|_\infty \Rightarrow \sup(\cdot) \ge \|\hat{G}\|_\infty$

$|\hat{U}_\varepsilon(j\omega)| = \begin{cases} \sqrt{\pi/\varepsilon}, & \omega_0 - \varepsilon \le \omega \le \omega_0 + \varepsilon \\ 0 & \text{o.w.} \end{cases}$

where $\|\hat{G}\|_\infty = \|\hat{G}(j\omega_0)\|$

$\|u_\varepsilon\|_2^2 = \|\hat{u}_\varepsilon(s)\|_2^2 = 1$

$\|\hat{Y}_\varepsilon(s)\|_2^2 = \frac{1}{2\pi}\int_{\omega_0-\varepsilon}^{\omega_0+\varepsilon} |\hat{G}(j\omega)|^2 \left(\sqrt{\frac{\pi}{\varepsilon}}\right)^2 d\omega$ — $\left(\lim_{\varepsilon\to 0}\right)$

$= \frac{1}{2\pi}\int_{\omega_0-\varepsilon}^{\omega_0+\varepsilon} |\hat{G}(j\omega_0)|^2 \pi/\varepsilon \, d\omega$

$= \|\hat{G}\|_\infty^2 \frac{1}{2\pi}\int_{\omega_0-\varepsilon}^{\omega_0+\varepsilon} \pi/\varepsilon \, d\omega$

$= \|\hat{G}\|_\infty^2 \|u_\varepsilon\|_2^2$

10/7 L3:

$u \in \mathcal{U}$ $y \in \mathcal{Y}$
$x \in \Sigma$ $t \in T$

$y(t) = p(t, t_0, x_0, u(\cdot))$

$= r(t, s(t, t_0, x_0, u(\cdot)), u(t))$

if linear $\begin{cases} = r(t, s(t, t_0, x_0, 0), 0) + r(t, s(t, t_0, 0, u(\cdot)), u(t)) \\ \quad \uparrow \text{linear w.r.t. } x_0 \qquad \uparrow \text{linear w.r.t. } u(\cdot) \\ = \text{zero-input response + zero-state response} \end{cases}$

$\|P\|_{2,2} = \sup_{\|u\|_2 \le 1} \|y\|_2$

Linear, state space, I/O differential equations

$\frac{dx}{dt} = f(x, u(t), t)$ $\quad y = h(x, t)$

$x \in \mathbb{R}^n = \Sigma$; $f: \mathbb{R}^n \times \mathbb{R}^m \times T \to \mathbb{R}^n$; $h: \mathbb{R}^n \times T \to \mathbb{R}^p$ — # of outputs

Linear Systems:

$dx/dt = A(t)x + B(t)u(t)$; $y = C(t)x$

→ Linear, time varying (non-autonomous) differential equations

$A(t) \in \mathbb{R}^{n\times n}$; $B(t) \in \mathbb{R}^{n\times m}$; $C(t) \in \mathbb{R}^{p\times n}$

→ Linear, time invariant (LTI) I/O system

$\dot{x} = Ax + Bu$; $y = Cx$

→ Even more specific: $u = 0$; $\dot{x} = Ax$; $y = Cx$

$x(t) = e^{A(t-t_0)} x_0 = s(t, t_0, x_0, 0)$

$y(t) = Ce^{A(t-t_0)} x_0 = p(t, t_0, x_0, 0)$

Matrix Exponential: $X \in \mathbb{R}^{n\times n}$; $e^X \in \mathbb{R}^{n\times n}$

$e^X = I + X + \frac{1}{2}X^2 + \frac{1}{3!}X^3 + \dots$

Ex: $A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$ $\quad \dot{x}_1 = -x_1$, $\dot{x}_2 = -2x_2$ $\mapsto$ $x_1(t) = e^{-(t-t_0)} x_1(0)$, $x_2(t) = e^{-2(t-t_0)} x_2(0)$

$\therefore e^{A(t-t_0)} = \begin{bmatrix} e^{-(t-t_0)} & 0 \\ 0 & e^{-2(t-t_0)} \end{bmatrix}$ $\quad t_0 = 0$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -t & 0 \\ 0 & -2t \end{bmatrix} + \begin{bmatrix} -\frac{1}{2}t^2 & 0 \\ 0 & -\frac{1}{2}(2t)^2 \end{bmatrix} + \dots$

Ex: $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ $\quad e^{At} = e^{at}\begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$

General:

-Transforming Coordinates:

$\dot{x} = Ax$; $z = Tx$; $T \in \mathbb{R}^{n\times n}$ (invertible)

$\dot{z} = T\dot{x} = TAx = TAT^{-1}z$

$t_0 = 0$ $\quad z(t) = e^{(TAT^{-1})t} z(0)$

$e^{(TAT^{-1})t} = I + (TAT^{-1})t + \frac{1}{2}(TAT^{-1})^2 t^2 + \frac{1}{3!}(TAT^{-1})^3 t^3$

$= I + (TAT^{-1})t + \frac{1}{2}(TAT^{-1})(TAT^{-1})t^2 + \dots$

$= I + (TAT^{-1})t + \frac{1}{2}TA^2T^{-1}t^2 + \frac{1}{3!}TA^3T^{-1}t^3$

$= T(I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \dots)T^{-1}$

$\therefore e^{(TAT^{-1})t} = Te^{At}T^{-1}$

$\therefore e^{At} = T^{-1}e^{(TAT^{-1})t}T$

~Suppose we find T s.t. TAT^{-1} is easy to calculate

Ex: Diagonalizable matrix

Ex: Jordan form

$J = \begin{bmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_k \end{bmatrix}$ $\quad J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}$

Ex: $\left[\begin{array}{cc|c|cc} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & -2 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right]$

10/7 **L3 cont'd:**

$$e^J = \begin{bmatrix} e^{J_1} & & 0 \\ & \ddots & \\ 0 & & e^{J_N} \end{bmatrix} \quad \begin{bmatrix} \lambda_k & 1 & 0 \\ & \ddots & 1 \\ 0 & & \lambda_k \end{bmatrix} = \underbrace{\begin{bmatrix} \lambda_k & & 0 \\ & \ddots & \\ 0 & & \lambda_k \end{bmatrix}}_{S_k} + \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ & \ddots & 1 \\ 0 & & 0 \end{bmatrix}}_{N_k}$$

Note: $SN = NS$

$$e^{A+B} = I + (A+B) + \tfrac{1}{2}(A+B)^2 + \cdots$$
$$= I + A + B + \tfrac{1}{2}(A^2 + AB + BA + B^2) + \cdots$$
$$+ \tfrac{1}{2}(A^2 + 2AB + B^2) + \cdots$$

$\cdots e^{A+B} = e^A e^B$ ★Good to do on own

$$\therefore e^{J_k} = e^{S_k} e^{N_k}$$
$$= \begin{bmatrix} e^{\lambda_k} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_k} \end{bmatrix} \left(I + N_k + \tfrac{1}{2}N_k^2 + \cdots + \tfrac{1}{(m_k-1)!} N_k^{m_k-1}\right)$$

$$N_k = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & 0 \end{bmatrix}; \quad N_k^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ & \ddots & \ddots & 1 \\ & & & 0 \\ 0 & & & 0 \end{bmatrix}$$

$$e^{J_k t} = \begin{bmatrix} e^{\lambda_k t} & te^{\lambda_k t} & \cdots & \frac{1}{(m_k-1)!} t^{m_k-1} e^{\lambda_k t} \\ & e^{\lambda_k t} & & \vdots \\ & & \ddots & \\ 0 & & & e^{\lambda_k t} \end{bmatrix}$$

Theorem: $\forall$ matrix $A \in \mathbb{R}^{n\times n}$ $\exists$ $T \in \mathbb{R}^{n \times n}$ s.t. TAT^{-1} is in Jordan form

Claim: Sol'n for $\dot{x} = Ax$, $x(0) = x_0$

$$x(t) = e^{At} x(0) \quad \rightsquigarrow e^{At} = T^{-1} e^{Jt} T$$

Proof. $\frac{dx}{dt} = \frac{d}{dt}\left(I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots\right)x(0)$

$$\left(0 + A + A^2 t + \tfrac{1}{2}A^3 t^2 + \cdots\right) x(0)$$
$$= A(I + At + \cdots)x_0 = Ae^{At}x_0 \quad \blacksquare$$

$$y(t) = r(t, \underbrace{s(t, t_0, x_0, 0)}_{Ce^{A(t-t_0)}x_0}, 0) + r(t, s(t, t_0, x_0, u(\cdot))u(t))?$$

Claim: If $x(0) = 0$ then $x(t) = \int_0^t e^{A(t-\tau)} Bu(\tau)\,d\tau$

$$y(t) = \int_0^t \underbrace{Ce^{A(t-\tau)}B}_{h(t-\tau)} u(\tau)\,d\tau$$

~Proof in notes.

Complex version of Jordan form:

$$J_k = \left[\begin{array}{c|c|c} \begin{matrix} a_k & -b_k \\ b_k & a_k \end{matrix} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 0 \\ \hline 0 & \ddots & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \\ \hline 0 & 0 & \begin{matrix} a_k & -b_k \\ b_k & a_k \end{matrix} \end{array}\right]$$

$\lambda_k = a_k + ib_k$ if A has complex eigenvalues.

Time varying systems: $\Phi(t, t_0) = e^{A(t-t_0)}$

$$y(t) = C^{(t)}\Phi(t, t_0)x(0) + \int_0^t C^{(t)}\Phi(t, \tau)B^{(t)}u(\tau)\,d\tau$$

$\Phi(t, t_0)$ is the fundamental matrix

$$\tfrac{d}{dt}\Phi(t, t_0) = A^{(t)}\Phi(t, t_0) \qquad \Phi(t_0, t_0) = I$$

10/9 **L4: Stability**

I/O bounds: $\|y\|_2 \le \|G\|_\infty \|u\|_2$

I/O "stability": bounded input $\Rightarrow$ bounded output

$\dot{x} = Ax \rightsquigarrow x(t) = e^{At}x(0)$

If all eigenvalues λ_k of A have $\mathrm{Re}(\lambda_k) < 0$

$\Rightarrow e^{At}x(0) \to 0$ as $t \to \infty$

Invariant Subspaces:

Definition: A subspace $V \subset \mathbb{R}^n$ of dim $\ell \le n$ is *A-invariant* if $AV \subset V$ and is *e^{At}-invariant* if $e^{At}V \subset V$ $\forall t$.

Claim: All A-invariant subspaces are e^{At}-invariant

What are the A-invariant subspaces for any A?

1) Any eigenvector v defines an A invariant subspace $\subset \mathrm{span}\{v\}$
$$Av = \lambda v \in \mathrm{span}\{v\}$$

2) Any $V = \mathrm{span}\{v_1, \ldots, v_p\}$ where v_i is a λ-vector
$$A(\alpha_1 v_1 + \cdots + \alpha_p v_p) = \alpha_1 Av_1 + \cdots + \alpha_p A_p$$

$\to A = TJT^{-1}$ $\quad V_{jk}$ is J-inv

$$TV_{jk} = \mathrm{span}\{Tv_{jk,1}, \ldots, Tv_{jk,m_k}\}$$
$$A(TV_{jk}) = TJT^{-1}T \cdot V_{jk}$$
$$= T\underbrace{JV_{jk}}_{\subset V_{jk}}$$

$\rightsquigarrow$ take block elements of jordan form

Special invariant subspaces $\qquad \mathrm{Re}(\lambda_k) < 0$

E^s = invariant subspace corresponding to all "stable" eigenvalues

Ex:
$$J = \left[\begin{array}{c|c|c|c} \begin{matrix} -1 & 1 \\ & -1 \end{matrix} & & & \\ \hline & -2 & & \\ \hline & & \begin{matrix} 0 & 1 \\ & 0 \end{matrix} & \\ \hline & & & 3 \end{array}\right] \qquad \lambda(TJT^{-1}) = \lambda(J)$$

$$E^s = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

E^u = subspace corresponding to $\mathrm{Re}(\lambda) > 0$

E^c = — " — $\qquad \mathrm{Re}(\lambda) = 0$ ~ (c stands for center)

$X(0) = X_0^s + X_0^u + X_0^c$ $\rightsquigarrow$ all orthogonal subspaces b/c of jordan form

$$X(t) = \underbrace{e^{At}X_0^s}_{\substack{\subset E^s \\ \to 0}} + \underbrace{e^{At}X_0^u}_{\substack{\subset E^u \\ \to \infty}} + \underbrace{e^{At}X_0^c}_{\substack{\text{not} \to 0 \\ \text{may be unbounded}}}$$

ex: $\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \rightsquigarrow e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ $\rightsquigarrow$ unbounded

10/9 L4 cont'd:

More generally: Autonomous nonlinear diff eq.

$\dot{x} = f(x)$

Eq pt x_e is any point for which $f(x_e)=0$

$e^{At}x_0 = 0 \to \dot{x} = Ax \to Ax_e = 0$

↳ Null space of A

An eq. pt. x_e is stable if for any $\varepsilon \; \exists \, \delta$ s.t.

for any x_0 satisfying $\|x(0) - x_e\| < \delta \Rightarrow \|x(t) - x_e\| < \varepsilon$

start near $\Rightarrow$ stay near

An eq. pt. x_e is asymptotically stable if it is stable and $\|x(t) - x_e\| \to 0$ as $t \to \infty$

Linear system $x_e = 0$ is always an eq. pt.

Asympt. stable $\Leftrightarrow Re(\lambda) < 0 \;\; \forall \lambda$ e-values of A

~~stable if only jordan blocks~~

Must be careful with non-trivial Jordan blocks

Eq. pt. is exponentially stable if $\exists$ constants $\alpha, \beta, m, M > 0$ s.t. (assuming $x_e = 0 \to$ can be reduced to this)

$\|x_0\| m e^{-\alpha t} \leq \|x(t)\| \leq M e^{-\beta t} \|x_0\|$

↑ most negative eigenvalue

↳ can't converge faster for linear system

If $Re(\lambda) < 0$ & trivial Jordan blocks then $\alpha = \min Re(\lambda_n) < 0$

$\beta = \max Re(\lambda_n) < 0$

→ envelope of decay → can overshoot within bounds.

Linear: BIBO stable $\Leftrightarrow$ exponentially stable

10/11 Recitation Coordinate Transformations

$\left.\begin{array}{l}\dot{x} = Ax + Bu \\ y = Cx + Du\end{array}\right\}$ is this unique? $\exists (A_1, B_1, C_1, D_1)$ where behavior is identical?

$x \in \mathbb{R}^n$, $T \in \mathbb{R}^{n \times n}$ invertible

$z = Tx \;\; \leadsto \;\; x = T^{-1}z$

$\dot{z} = T\dot{x} = T(Ax + Bu)$

$\dot{z} = \underbrace{(TAT^{-1})}_{\hat{A}} z + \underbrace{TB}_{\hat{B}} u = \hat{A}z + \hat{B}u = \dot{z}$

$y = \underbrace{CT^{-1}}_{\tilde{C}} z + Du = \tilde{C}z + Du$

Linear systems are coordinate transform invariant.

$\therefore A$ and $\hat{A}$ have identical Eigenvalues/vectors

→ Poles are identical

To show this: $Av = \lambda v$; $\hat{A}\tilde{v} = \tilde{\lambda}\tilde{v}$

$(TAT^{-1})\tilde{v} = \tilde{\lambda}\tilde{v}$

$A(T^{-1}\tilde{v}) = \tilde{\lambda}(T^{-1}\tilde{v}) \;\; \to \;\; \lambda(A) = \lambda(\hat{A})$

$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$

$z(t) = e^{\hat{A}t}z_0 + \int_0^t e^{\hat{A}(t-\tau)}\hat{B}u(\tau)\,d\tau$

$z = Tx \qquad \hat{A} = TAT^{-1} \qquad \hat{B} = TB \qquad z_0 = Tx_0$

(*) $e^{TST^{-1}} = Te^{S}T^{-1}$

$z(t) = e^{(TAT^{-1})t}Tx_0 + \int_0^t e^{TAT^{-1}(t-\tau)}TB\,u(\tau)d\tau$

$= Te^{At}x_0 + T\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$

$z(t) = Tx(t)$

Steady-state response → step response

$\rho(t, t_0, x_0, u(\cdot)) = \rho(t, t_0, 0, u(\cdot)) + \rho(t, t_0, x_0, 0)$

$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$

forced response: transience → steady state

$x(0) = 0$

$y(t) = \int_0^t Ce^{A(t-\tau)}Bu(\tau)\,d\tau + D$ → step response.

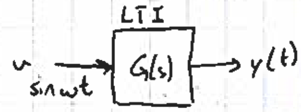
$= C\left(\int_0^t e^{A(t-\tau)}d\tau\right)B + D$

$= CA^{-1}e^{At}B - CA^{-1}B + D$

$y(t) = \underbrace{(CA^{-1}e^{At}B)}_{\text{transience}} + \underbrace{(D - CA^{-1}B)}_{\text{steady state}}$

→ can do same thing for impulse response

Ex:



$\dot{x} = Ax + Bu$
$y = Cx + Du$
$\lambda = \text{eig}(A)$

$x(0) = 0$

• $\omega >> |\lambda|$

start with $u = e^{st}$

$y(t) = C\int_0^t e^{A(t-\tau)}Be^{s\tau}d\tau$

$= Ce^{At}\int_0^t e^{(sI-A)\tau}B\,d\tau$

$= Ce^{At}(sI-A)^{-1}(e^{(sI-A)t} - I)B$

$= C(sI-A)^{-1}Be^{st} - C(sI-A)^{-1}e^{At}B$

$s = j\omega \qquad \omega >> \lambda(A)$

$(j\omega I - A)^{-1} \simeq -\frac{j}{\omega}$

$y(t) = G(j\omega)e^{j\omega t} - C(j\omega I - A)^{-1}e^{At}B$

$\simeq G(j\omega)e^{j\omega t} + \frac{j}{\omega}h(t)$ $\qquad h(t)$ is impulse response

$u_{real}(t) =$ ~~cos(ωt) + j sin(ωt)~~ $\cos(\omega t) + j\sin(\omega t)$

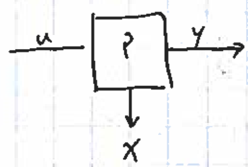
$y(t) = \mathrm{Im}(\;\to\;)$

$y(t) = |G(j\omega)|\sin(\omega t + \arg(G(j\omega))) + \frac{1}{\omega}h(t)$

10/14 L5:

$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau$

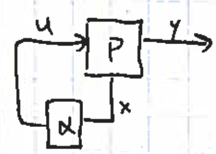
stability $\leftrightarrow \lambda(A) \leadsto E^s, E^u, E^c$



Can we choose $u_{[0,T]}$ s.t. if $x(0) = x_0$ then $x(T) = x_f$ → given

$x_0 \underset{T}{\leadsto} x_f$

Reachability



$\dot{x} = f(x,u) \qquad x(0) = x_0$

$u = \alpha(x)$

Q: Is $\dot{x} = f(x, \alpha(x))$ stable?

Stabilizability

General concepts: $D = \{\mathcal{U}, \Sigma, \mathcal{Y}, s, r\}$

Definition: the state $x_f \in \Sigma$ is reachable from $x_0 \in \Sigma$ in time $T > 0$ if $\exists \, u \in \mathcal{U}_{[0,T]}$ s.t. $x(T) = x_f$ ($x_0 \underset{T}{\leadsto} x_f$ is true)

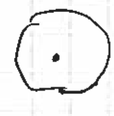
Reachable set in some time

$R_{\leq T}(x_0) \equiv R(x_0, \leq T) : \{x(\tau) : u \in \mathcal{U}_{[0,T]}, x(0) = x_0 \text{ and } \tau \leq T\}$

Definition: A system D is small-time locally controllable (STLC) at x_0 if for every $T > 0 \;\; \exists \, \varepsilon > 0 \;\; R(x_0, \leq T) \supset B_\varepsilon(x_0) = \{x : \|x - x_0\| < \varepsilon\}$

Ex: Reachable:

STLC



$R(x_0 \leq T)$

10/14 L5 cont'd:

Linear System: $\dot{x} = Ax + Bu$ $x \in \mathbb{R}^n, u \in \mathbb{R}^m$

To check reachability, look at range of linear map $\mathcal{L}_T : \mathcal{U} \to \mathbb{R}^n \rightsquigarrow \mathcal{L}_T(u(\cdot)) = \int_0^T e^{A(T-\tau)} Bu(\tau)\,d\tau$

Don't include $e^{At}x_0$ b/c $\xrightarrow{T} e^{AT}x_0 \cdots \tilde{x}_f = x_f - e^{AT}x_0$ can be represented as constant offset (drift) on x_f

$L: \mathbb{R}^q \to \mathbb{R}^n$ $q > n$; $x_f = Lu$

$x_f = \boxed{\quad L \quad} u$ need rank $L = n$

$u = L^T(LL^T)^{-1}x_f$

↳ least squares u, can also have u in null space

Instead of L^T write L^* (adjoint matrix → in finite dimensions $L^* = L^T$)

$$(L_T^* w)(t) = \begin{cases} B^T e^{A^T(T-t)} w & \text{if } t \le T \\ 0 & \text{if } t > T \end{cases}$$

~~Claim~~ Condition: need rank = dim(range $\mathcal{L}_T$) = n

Claim: dim(range $\mathcal{L}_T$) = dim range($\mathcal{L}_T \mathcal{L}_T^*$)

$$\mathcal{L}_T \mathcal{L}_T^* = \int_0^T e^{A(T-\tau)} BB^T e^{A^T(T-\tau)}\,d\tau$$

$$\begin{bmatrix}\text{Controllability Gramian} \\ \text{for time } T\end{bmatrix} = \int_0^T e^{A\tau} BB^T e^{A^T\tau}\,d\tau = W_c(T)$$

~~Theorem: A linear system is reachable~~

Given $x_0 = 0$ & x_f claim that

$$u(t) = B^T e^{A^T(T-\tau)} W_c^{-1}(T) x_f$$

steers $0 \underset{T}{\rightsquigarrow} x_f$.

Claim 1: If $W_c(T)$ has rank n, we can go b/w any two points in time T (Proof: plug in)

Claim 2: The dimension of the range ("rank") of $W_c(T)$ is independent of $T > 0$

Proof: (sketch) Suppose not full rank

Then $\exists$ $v \in \mathbb{R}^n$ s.t. $v^T W_c(T) = 0$

$\int_0^T v^T e^{A\tau} B^T e^{A^T\tau}\,d\tau = 0$

↳ v can go on either side. Will make it zero for all time. ∴ Reachability is time independent

If A is stable then let $T \to \infty$.

Define $W_c = \lim_{T\to\infty} W_c(T) = \int_0^\infty e^{A\tau} BB^T e^{A^T\tau}\,d\tau$

Theorem: A linear system is reachable iff $W_c \succ 0$

Cor: Linear system is STLC iff $W_c \succ 0$

Even Better!

1) $AW_c + W_c A^T = BB^T$

2) $\|G\|_2 = \|\hat{G}\|_2 = \sqrt{CW_cC^T}$

2 norm → ∞ norm gain

10/16 L6:

$W_c(T) = \int_0^T e^{A^*\tau} B^* B e^{A^T\tau}\,d\tau$

$u(t) = B^T e^{A^T(T-t)} W_c^{-1}(T) x_f$

Theorem: An LTI system is reachable (for any x_0, x_f, T) and STLC iff any of the following hold:

(1) $W_c(T) \succ 0$ for any $T > 0$

(2) W_c satisfying $AW_c + W_c^* A^T = -BB^T$ satisfies $W_c \succ 0$ (and $W_c = \lim_{T\to\infty} W_c(T)$ for A stable)

(3) rank $\underbrace{[sI - A \mid B]}_{n \times m}\}n = n$ $\forall s \in \mathbb{C}$ → really only need to check $s = \lambda(A)$

4) rank $[B \mid AB \mid A^2B \mid \cdots \mid A^{n-1}B] = n$

Proof: (#2 → rest in lecture notes)

$AW_c + W_cA^T = A\left(\int_0^\infty e^{A\tau} BB^T e^{A^T\tau}\,d\tau\right) + \left(\int_0^\infty e^{A\tau} BB^T e^{A^T\tau}\,d\tau\right)A^T$

$= \int_0^\infty \frac{d}{d\tau}\left(e^{A\tau} BB^T e^{A^T\tau}\right)d\tau$

$= \left(e^{At} BB^T e^{A^Tt}\right)\Big|_{t=0}^{t=\infty} = -BB^T$

Suppose that W_c is not full rank

$\Leftrightarrow \exists$ v s.t. $v^T W_c v = 0$

v represents an unreachable direction

$v^T e^{A\tau} BB^T e^{A^T\tau} v = 0 \equiv v^T e^{A\tau} B = 0 \;\forall \tau$

$\Rightarrow \int_0^t e^{A(t-\tau)} Bu(\tau)\,d\tau$ → zero when projected onto v.

↳ also true at $\tau = 0$ → ∴ $v^T B = 0$

also $\frac{d}{d\tau}(v^T e^{A\tau} B) = v^T A e^{A\tau} B = 0 \xrightarrow{} v^T AB = 0$

↳ leads to #4

#3) PBH test → dropped rank means

$v^T[sI - A] = 0$ → $v^T B = 0$, $v^T(sI - A) = 0$

v^T must be a left eigenvector of A (e.v. of A^T)

On all tests: If we drop rank on any above test, ($\exists v$) we are trying to find directions you can't go.

Reachable space (from origin) $\mathcal{R}(0, \le T)$

= largest A invariant subspace containing B

$v^T W_c = 0$, v^T unreachable

$\mathcal{R} = \{w : v^T w = 0\}$

Stability: $\dot{x} = Ax + Bu$ $u = -kx$ (A, B) is stabilizable

Want $\dot{x} = (A - Bk)x$ asy stable $\text{Re}(A - Bk) < 0$

Stronger condition: given desired eigenvalues $\lambda_1^d, \ldots, \lambda_n^d$ find k s.t. $\lambda(A - Bk) = \{\lambda_1^d, \ldots, \lambda_n^d\}$

↖ Eigenvalue (Pole) placement ⟺ STLC

To prove, find T s.t. (coefficients of char. eq.)

$$TAT^{-1} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & & & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{bmatrix} \quad T^{-1}B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

If system is reachable such a T exists. (FBS 2e)

$\det(sI-A) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1}s + a_n$

$$(A-Bk) = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & \ddots & & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0\end{bmatrix} - \begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix}[k_1, k_2 \cdots k_n]$$

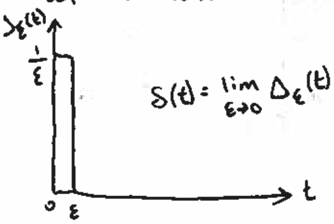
$[u = -kx]$

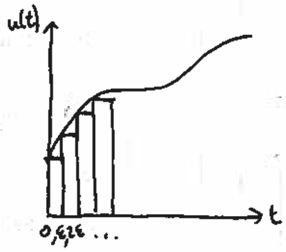
$$= \begin{bmatrix} -(a_1+k_1) & -(a_2+k_2) & \cdots & -(a_n+k_n) \\ 1 & & & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{bmatrix}$$ ← roots of feedback char. poly.

→ Pole placement.

10/18 Recitation:

Proof of convolution:





$u(t) = \lim_{\varepsilon\to0}\sum_{i=0}^\infty u(i\varepsilon)\,\varepsilon\,\Delta_\varepsilon(t-i\varepsilon)$

Impulse response $h(t) = \lim_{\varepsilon\to0} y_{\Delta\varepsilon}(t)$; $y_{\Delta\varepsilon}(t) = \mathcal{L}(\Delta_\varepsilon(t))$

$\mathcal{L}(u(t)) = \lim_{\varepsilon\to0}\{\sum_{i=0}^\infty u(i\varepsilon)\,\varepsilon\, \mathcal{L}[\Delta_\varepsilon(t-i\varepsilon)]\}$

$y(t) = \lim_{\varepsilon\to0}\left[\sum_{i=0}^\infty u(i\varepsilon)\,\varepsilon\, y_{\Delta\varepsilon}(t-i\varepsilon)\right] = \int_0^\infty u(\tau)h(t-\tau)\,d\tau$

10/21 L7: Optimal Control

Consider the system $\dot x = f(x,u)$ with $u \in \underset{\text{constrained}}{\mathcal{U}} \subset \mathbb{R}^m$

Find $u(\cdot)$ such that: ← terminal cost

$\min \int_0^T L(x(\tau),u(\tau))\,d\tau + V(x(T))$

s.t. $\dot x = f(x,u)$ $\quad x(0) = x_0$

$u \in \mathcal{U}$ $\quad \Psi(x(T)) = 0 \longrightarrow \Psi = \begin{bmatrix}\Psi_1(x)\\ \vdots \\ \Psi_q(x)\end{bmatrix}$

← final constraint $\quad 0 \le q \le n$

(e.g. $\|x(T) - x_f\|^2 = \Psi(x(T))$)

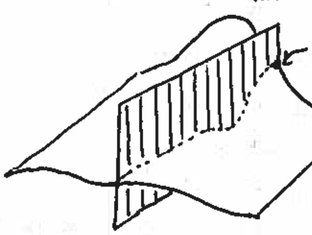
[We will eventually specialize to
$\dot x = Ax + Bu \quad u\in\mathbb{R}^m$
$J = \int_0^T (x^TQx + u^TRu)\,d\tau + x^T(T)P_1x(T)$
$Q \ge 0 \quad R > 0 \quad$ (LQR)]

Motivate/Review Static optimization

$F: \mathbb{R}^n \to \mathbb{R}$, optimal input → find x^* s.t. $F(x^*) \ge F(x)$ $\forall x\in\mathbb{R}^m$

Necessary condition: $\frac{\partial F}{\partial x}(x^*) = 0$

Optimization with constraints: −〃− subject to $G(x) = 0$



← constrained optimal solution → $\frac{\partial F}{\partial x}$ not necessarily zero

∴ Consider only in-plane gradient of F

$\frac{\partial F}{\partial x} + \sum \lambda_i^* \frac{\partial G}{\partial x} = 0$ for some $\lambda\in\mathbb{R}^k$

Special Cases:

1. LQR
2. $T=\infty$: infinite horizon (drop $V(x(T))$ & $\Psi(x(T))$)
3. Constrained $\mathcal{U}$

Define: Hamiltonian $H: \mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^n \to \mathbb{R}$

[A] $H(x,u,\lambda) = L(x,u) + \lambda^T f(x,u) = L(x,u) + \sum \lambda_i f_i(x,u)$

Theorem: Maximum Principle: If (x^*, u^*) is optimal, then $\exists\, \lambda^*(t)\in\mathbb{R}^n$ and $\nu^*\in\mathbb{R}^q$

(a) $\dot x_i^* = \partial H/\partial\lambda_i (x^*,u^*,\lambda^*)$ $\quad x(0)$ given $\quad \Psi(x^*(T)) = 0$

(b) $\dot\lambda_i^* = -\partial H/\partial x_i (x^*,u^*,\lambda^*)$ $\quad \lambda^*(T) = \frac{\partial V}{\partial x}(x^*(T)) + \nu^{*T}\frac{\partial\Psi}{\partial x}$

and (c) $H(x^*(t),u^*(t),\lambda^*(t)) \le H(x^*(t),u,\lambda^*(t))$ $\forall\, u\in\mathcal{U} \sim \Omega$ in notes

Structure:

(a) $\partial H/\partial\lambda = f(x,u)$ $\quad x\in\mathbb{R}^n$

(b) $\lambda\in\mathbb{R}^n$ Just an ODE with terminal condition

(c) Find $u^*\in\mathbb{R}^m$ s.t. if $\Omega = \mathbb{R}^m$ then a necessary condition is $\partial H/\partial u (x^*,u^*,\lambda^*) = 0$

Finding optimal trajectories:

Step 1: Given x,λ, find minimum u where $u = u(x,\lambda)$

Step 2: Solve the two-point boundary value problem

$\dot x = f(x,u(x,\lambda))$ $\quad x(0) = x_0$

$\dot\lambda = -\frac{\partial H}{\partial x_i}(x,u(x,\lambda),\lambda)$ $\quad \lambda(T) =$ something $\left[\frac{\partial V}{\partial x}(x(T)) + \nu^T\frac{\partial\Psi}{\partial x}\right]$ ©

Suppose $L(x,u) = x^TQx + u^TRu$ $\quad R>0$

$\Psi(x(T)) = x^TP_1x$ $\quad \dot x = Ax+Bu$

$\dot x = Ax + Bu \rightsquigarrow \dot x = Ax - BR^{-1}B^T\lambda$ $\quad x(0) = x_0$

$\dot\lambda = -Qx + A^T\lambda$ $\quad \lambda(T) = P_1x(T)$ $\quad B^T\lambda$

$0 = \partial H/\partial u = Ru + \lambda^TB \rightsquigarrow u = -R^{-1}\lambda^TB$

Let's guess that $\lambda(t) = P(t)x(t)$

Plug into $\dot\lambda \to -\dot Px - PAx + PBR^{-1}B^TPx = Qx + A^TPx$

$-\dot P = PA + A^TP - PBR^{-1}B^TP + Q$ ← → $P(T) = P_1$ $\quad$ Riccati ODE

Strategy:

- Solve $P(t)$ by reverse integrating
- set $\lambda(t) = P(t)x(t)$
- solve $\dot x = Ax - BR^{-1}B^TP(t)x$ $\quad x(0) = x_0$ by integrating forward in time.

10/23 L8:

- From before, $u^* = \arg\min_u H(x,u,\lambda) \rightsquigarrow u(t) = -R^{-1}B^TP(t)x$

Remarks: 1) Can change from "steering" to origin to steering to x_f eq. pt.

$z = x - x_f$; $\dot z = Ax + Bu = A(z+\cancel{x_f}) + Bu$; $\dot z = Az + Bu$

2) Control is in the form of feedback law

3) If $T = \infty$ and $P_1 = 0$ then if $\exists$ a constant P satisfying

$0 = PA + A^TP - PBR^{-1}B^TP + Q$ ← algebraic Riccati equation (ARE)

[If (A,B) is reachable then $P>0$ satisfying A.R.E. always exists]

→ then $u = \underbrace{-R^{-1}B^TP}_{K}x$ is optimal

Recall $\min F(x)$ s.t. $G_i(x) = 0$ $\quad \tilde F = F + \lambda^TG$, $\partial\tilde F/\partial x = 0$

$\tilde J(x,u,\lambda) = J(x,u) + \int_0^T -\lambda^T(t)(\dot x(t) - f(x,u))\,dt + \nu^T\Psi(x(T))$

$= \int_0^T (L(x,u) - \lambda^T(t)(\dot x(t) - f(x,u)))\,dt + V(x(T)) + \nu^T\Psi(x(T))$

$\tilde J(x(\cdot),u(\cdot),\lambda(\cdot)) = \int_0^T (H(x,u) - \lambda^T\dot x)\,dt + V(x(T)) + \nu^T\Psi(x(T))$

$\delta\tilde J = \tilde J(x^*+\delta x, u^*+\delta u, \lambda^*+\delta\lambda) - \tilde J(x^*,u^*,\lambda^*)$

$\simeq \int_0^T \frac{\partial H}{\partial x}\delta x + \frac{\partial H}{\partial u}\delta u - \lambda^T\delta\dot x + (\frac{\partial H}{\partial\lambda} - \dot x^T)\delta\lambda\,dt + \overbrace{\frac{\partial V}{\partial x}\delta x(T)}^{©} + \overbrace{\nu^T\frac{\partial\Psi}{\partial x}\delta x(T)}^{©} + \delta\nu^T\Psi(x(T),u(T))$

Integration by parts: $-\int_0^T \lambda^T\delta\dot x\,dt = \underbrace{-\lambda^T(T)\delta x(T)}_{©} + \cancel{\lambda^T(0)\delta x(0)} + \int_0^T \dot\lambda^T\delta x\,dt$

10/23 L8 cont'd:

Maximum principle:

$\dot x = f(x,u)$ $\quad x(0)=x_0$ $\quad u\in\Omega$

$J(x,u)=\int_0^T L(x,u)\,dt + V(x(T)) \to \min J(x,u)$ s.t. $\psi(x(T))=0$

$H(x,u,\lambda) = L(x,u) + \lambda^T f(x,u)$

optimal solution: $\quad x\in\mathbb{R}^n$, $\lambda\in\mathbb{R}^n$

$\dot x = \frac{\partial H}{\partial\lambda}(x,u,\lambda)$ $\quad x(0)=x_0$

$-\dot\lambda = \frac{\partial H}{\partial x}(x,u,\lambda)$ $\quad \lambda(T)=\frac{\partial V}{\partial x}(x(T)) + \nu^{*T}\frac{\partial \psi}{\partial x}$

$u^* = \arg\min_u H(x,u,\lambda)$; $\frac{\partial H}{\partial u}(x,u^*,\lambda)=0$

...All we are doing is $\min F(x)$ s.t. $G_i(x)=0$

$\tilde F = F+\lambda^T G$, $\partial\tilde F/\partial x = 0$ for ∞ dimensional spaces

Choosing Cost Function:

Most Common: $Q=I$; $R=\rho I$

$J=\int_0^\infty (\|x\|^2+\rho^2\|u\|^2)\,dt$

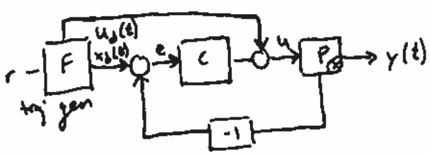
ρ trades error for energy

Next most common:

$$Q=\begin{bmatrix} q_{11} & & 0\\ & \ddots & \\ 0 & & q_{nn}\end{bmatrix} \quad R=\begin{bmatrix} r_{11} & & 0\\ & \ddots & \\ 0 & & r_{mm}\end{bmatrix}$$

$J=\int_0^\infty (q_{11}x_1^2+q_{22}x_2^2+\dots+q_{nn}x_n^2+u^TRu)\,dt$

Two degree-of-freedom design (for NL systems):



$\dot x = f(x,u)$ $\quad$ C can be $f(x_d,u_d)$

$-\dot x_d = f(x_d,u_d)$

$\dot e = f(x,u)-f(x_d,u_d) = f(x_d+e,\, u_d+v) - f(x_d,u_d)$

$= F(e,v,x_d,u_d) \simeq A(x_d,u_d)e + B(x_d,u_d)v$

10/25 Recitation: Discrete Time LQR

$x_{t+1}=Ax_t+Bu_t$; x_0 given

LQR (Objective Function)

$J(\mathcal U)=\sum_{t=0}^{N-1}(x_t^TQx_t+u_t^TRu_t)+x_f^TQ_fx_f$: $\mathcal U=(u_0,u_1,\dots,u_{N-1})$

$Q=Q^T\geq 0$, $Q_f=Q_f^T\geq0$, $R=R^T>0$

Aside: $u=\mathcal L_r^*(\mathcal L_r\mathcal L_r^*)^{-1}x_f$ minimizes $\|u\|_{L^2}$

Least Squares Solution:

$$\begin{bmatrix} x_0\\ x_1\\ \vdots\\ x_N\end{bmatrix} = \begin{bmatrix} 0 & & & 0\\ B & 0 & & \\ AB & B & \ddots & \\ \vdots & & \ddots & 0\\ A^{N-1}B & \cdots & & B\end{bmatrix}\cdot\begin{bmatrix} u_0\\ u_1\\ \vdots\\ u_{N-1}\end{bmatrix} + \begin{bmatrix} I\\ A\\ \vdots\\ A^N\end{bmatrix}x_0$$

$X = G\cdot\mathcal U + Hx_0$

cost: $J(\mathcal U)=\|\mathrm{diag}(Q^{1/2},\dots,Q^{1/2},Q_f^{1/2})\overbrace{(G\mathcal U+Hx_0)}^{X}\|^2 + \|\mathrm{diag}(R^{1/2},\dots,R^{1/2})\,\mathcal U\|^2$

DP: $J_t(\mathcal U_t)=\sum_{\tau=t}^{N-1}(x_\tau^TQx_\tau+u_\tau^TRu_\tau)+x_N^TQ_fx_N$

$(t<N)$ $\quad =(x_t^TQx_t+u_t^TRu_t) + \underbrace{\sum_{\tau=t+1}^{N-1}(x_\tau^TQx_\tau+u_\tau^TRu_\tau)+x_N^TQ_fx_N}_{J_{t+1}(\mathcal U_{t+1})}$

Define value function: $V_t(x_t) = \min_{\mathcal U_t} J_t(\mathcal U_t)$

$= \min_{u_t,\dots,u_{N-1}} \sum_{\tau=t}^{N-1}(x_\tau^TQx_\tau+u_\tau^TRu_\tau)+x_N^TQ_fx_N$

s.t. $x_t = x_t$
$x_{\tau+1}=Ax_\tau+B\tau$, $\tau=t,\dots,N-1$

Anzatz: $V_t(z)=z^TP_tz$

Proof:

① Base: $t=N$; $J_N(z)=x_N^TQ_fx_N \Rightarrow P_N=Q_f$

$V_N(z)=\min_u J_N(z)=z^TQ_fz$

② Induction: Assume $V_{k+1}(z)=z^TP_{k+1}z$

$V_k(z)=\min_{u_k,\dots,u_{N-1}} \sum_{\tau=k}^{N-1}(x_\tau^TQx_\tau+u_\tau^TRu_\tau)+x_N^TQ_fx_N$

$=\min_{u_k}\{x_k^TQx_k+u_k^TRu_k+\min_{u_{k+1},\dots,u_{N-1}}\{\sum_{\tau=k+1}^{N-1}(\dots)+x_N^TQ_fx_N\}\}$

$=\min_{u_k}\{x_k^TQx_k+u_k^TRu_k+x_{k+1}^TP_{k+1}x_{k+1}\}$

$x_{k+1}=Ax_k+Bu_k=Az+Bu_k$

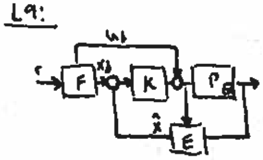
$=\min_{u_k}\{u_k^T(R+B^TP_{k+1}B)u_k+2u_k^TB^TP_{k+1}Az+z^T(Q+A^TP_{k+1}A)z\}$

$\frac{\partial V_k}{\partial u_k}=0\Rightarrow u_k^*=-(R+B^TP_{k+1}B)^{-1}BP_{k+1}Ax_k$

$V_k(z)=x_k^T\underbrace{[Q+A^TP_{k+1}A-A^TP_{k+1}B(R+B^TP_{k+1}B)^{-1}B^TP_{k+1}A]}_{P_k}x_k$ ■

Procedure:
① Set $P_N=Q_f$
② For $t=N,\dots,1$ $\quad P_{t-1}=Q+A^TP_tA-A^TP_tB(R+B^TP_tB)^{-1}B^TP_tA$
③ For $t=0,\dots,N-1$ $\quad K_t=-(R+B^TP_{t+1}B)^{-1}B^TP_{t+1}A$
④ For $t=0,\dots,N-1$ $\quad$ optimal $u_t^*=K_tx_t$

10/28 L9: $\quad$ Estimator: $\dot x=f(x,u)$ $\quad x\in\mathbb R^n$ $\quad u\in\mathbb R^m$
$y=h(x)$ $\quad y\in\mathbb R^q$



$\frac{d\hat x}{dt}=\underbrace{f(\hat x,u)}_{\text{prediction}}+\underbrace{\alpha(y-h(\hat x))}_{\text{correction}}$

Q1: When can we determine $x(t)$ from $y_{[0,t]}, u_{[0,t]}$

Distinguishability: $D=(\mathcal U,\Sigma,\mathcal Y,s,r)$

I/O map: $\rho(t,x_0,t_0,u(\cdot))=r(t,s(t,x_0,t_0,u),u)$

Definition: A dynamical system D is distinguishable on $[t_1,t_2]$ if $\forall\, x_0\neq z_0$ $\exists\, u(\cdot)$ s.t. $\rho(t,x_0,t_0,u(\cdot))\neq\rho(t,z_0,t_0,u(\cdot))$ for some $t\in[t_1,$

Definition: A dynamical system D is observable on $[t_1,t_2]$ if every $\forall u(\cdot)$, initial state x_0 is distinguishable for every $z_0\neq x_0$. D is observable if it is observable on any $[t_1,t_2]$. An observer is a mapping $O:\mathcal U\times\mathcal Y\to\Sigma$ that returns x_0 given $u(\cdot)$ & $y(\cdot)$.

$\rho(t,x_0,t_0,u(\cdot))=Ce^{At}+\int_0^tCe^{A(t-\tau)}Bu(\tau)\,d\tau$

$\rho:\Sigma\times\mathcal U\to\mathcal Y$

If $u(\cdot)\neq0$ then choose $\hat y = y-\int_0^t\cdots$ $\therefore$ W/O LOG assume $u(\cdot)=0$

$(\mathcal{M}_T(x_0))(t) = Ce^{At}x_0$ $\qquad \begin{bmatrix} M \end{bmatrix}\begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} y \end{bmatrix} \qquad x = (M^TM)^{-1}M^Ty$

$\mathcal{M}_T : \Sigma \to \mathcal{Y}$

$\mathcal{M}_T^* \sigma = \int_0^T e^{A^T\tau} C^T \sigma(\tau)\, d\tau$

$(\mathcal{M}_T^* \mathcal{M}_T) = \underbrace{\int_0^T e^{A^T\tau} C^T C e^{A\tau}\, d\tau}_{W_o(T)\ \to\ \text{observability Gramian}}$

(A,B) is observable iff $W_o(T)$ is full rank (n) and

$x_0 = (W_o(T))^{-1} \int_0^T e^{A^T\tau} C^T y(\tau)\, d\tau$

If (A,C) is not observable then $\exists\, v \neq 0 \in \mathbb{R}^n$ s.t. $W_o v = 0$ and all $x_0 = \alpha v$ w/ $\alpha \in \mathbb{R}$ generate identical outputs (0)

Can show: $W_o(T) > 0$ doesn't depend on T

If A is stable, define $W_o := \lim_{T\to\infty} W_o(T)$ and W_o satisfies

$A^T W_o + W_o A = -C^T C$

A pair (A,C) is observable iff the following conditions hold:

1) $W_o(T) > 0$ → + def ⇒ full rank, for some (or any) $T > 0$
2) If A is stable, solution of $A^T W_o + W_o A = -C^T C$ is pos. def (W_o = observability Gramian)
3) PBH: $\operatorname{rank}\begin{bmatrix} C \\ sI - A \end{bmatrix} = n \ \forall s \in \mathbb{C} \ (s \in \lambda(A))$
4) Observability rank test $\operatorname{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$

Estimators: "Observer" $u, y \to x_0$
"Estimator" $u, y \to x(t)$

$\dot{x} = Ax + Bu \qquad y = Cx$

$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \qquad e = \hat{x} - x$

$\dot{e} = Ae - LCe = (A - LC)e \qquad e(0) = \hat{x}(0) - x(0)$

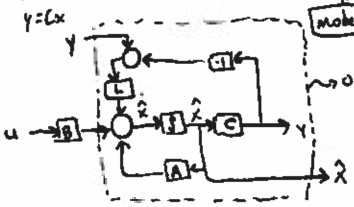
If $\lambda(A - LC)$ have $\operatorname{Re}\lambda < 0$ then $e \to 0$ as $t \to \infty$

Q: When can we assign λs to arbitrary values?

10/30 Recitation:

$\dot{x} = Ax + Bu$
$y = Cx$

Goal: drive error $= y - \hat{y}$ to 0. Do so by placing eigenvalues of L.



~ observer

In the above diagram

observer based controller design

$\hat{u} := -K\hat{x} \rightsquigarrow u = r + \hat{u}$

$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \ ; \ y = Cx \ ; \ \hat{y} = C\hat{x}$

$\Rightarrow \dot{\hat{x}} = A\hat{x} + Bu + LC(x - \hat{x}) \ ; \ \dot{x} = Ax + Bu$

$\tilde{x} = x - \hat{x} \ ; \ \dot{\tilde{x}} = \dot{x} - \dot{\hat{x}}$

$\dot{\tilde{x}} = A\tilde{x} - LC(x - \hat{x}) = \boxed{(A - LC)\tilde{x} = \dot{\tilde{x}}}$

Observer based state feedback controller: $u := -K\hat{x} + k_f r$

$\dot{x} = Ax + B(-K\hat{x} + k_f r)$ w/ $\hat{x} = x - \tilde{x}$

$\boxed{\dot{x} = (A - BK)x + BK\tilde{x} + Bk_f r}$

Now consider $\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix}\begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} Bk_f \\ 0 \end{bmatrix} r$

Separation principle: you can design the controller and observer independently.

- System should be reachable & controllable. If so you can arbitrarily place the eigenvalues of K & L independently.

Example: $\dot{x} = \begin{bmatrix} -4 & 1 \\ -6 & 1 \end{bmatrix} x + \begin{bmatrix} 3 \\ 7 \end{bmatrix} u \qquad \det(sI - A) = s^2 + 3s + 2$

$a_1 = 3, \ a_2 = 2$

$y = [1 \ \ -1] x$

$W_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \qquad \tilde{W}_o = \begin{bmatrix} 1 & 0 \\ -a_1 & 1 \end{bmatrix}$

↳ full rank $\therefore \exists\, T$ to $\tilde{x}, \tilde{A}, \tilde{B}$

$T = \tilde{W}_o^{-1} W_o = \begin{bmatrix} 1 & 1 \\ 5 & -3 \end{bmatrix}$

$\tilde{A} = TAT^{-1} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} \qquad \tilde{B} = TB = \begin{bmatrix} -4 \\ -6 \end{bmatrix} \qquad \tilde{C} = CT^{-1} = [1 \ \ 0]$

11/1 L10: Summary: Linear I/O system $\dot{x} = Ax + Bu$, $y = Cx$ $\quad x(0) = x_0 \quad u \in \mathbb{R}^m \quad x \in \mathbb{R}^n \quad y \in \mathbb{R}^q$

$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)} Bu(\tau)\, d\tau$

Reachability: $\mathcal{L}_T(u(\cdot)) = \int_0^T e^{A(T-\tau)} Bu(\tau)\, d\tau$

$W_c(T) = \int_0^T e^{A\tau} BB^T e^{A^T\tau}\, d\tau$

$AW_c + W_c A^T = -BB^T \qquad W_c = \lim_{t\to\infty} W_c(T)$

$u(t) = B^T e^{A^T(T-t)} W_c^{-1}(T) x_f$

Reachable set = range W_c

Observability: $\mathcal{M}_T(x_0) := Ce^{At}x_0$

$W_o(T) = \int_0^T e^{A^T\tau} C^T C e^{A\tau}\, d\tau$

$A^T W_o + W_o A = -C^T C$

$x_0 = (W_o(T))^{-1} \int_0^T e^{A^T\tau} C^T y(\tau)\, d\tau$

$\dot{x} = Ax + Bu, \ y = Cx$ what if not reachable and/or not observable?

Kalman Decomposition: $\mathbb{R}^n = E_{ro} \oplus E_{r\bar{o}} \oplus E_{\bar{r}o} \oplus E_{\bar{r}\bar{o}}$

$\left.\begin{array}{l} E_r = E_{ro} \oplus E_{r\bar{o}} = \text{range } W_c \\ E_{\bar{o}} = E_{r\bar{o}} \oplus E_{\bar{r}\bar{o}} = \text{null } W_o \end{array}\right\}$ Note: both are A-invariant

$E_{r\bar{o}} = \text{range } W_c \cap \text{null } W_o$ ← A-invariant

E_{ro} = space such that $E_r = E_{ro} \oplus E_{r\bar{o}}$ ← not necessarily A-invariant
↑ not unique, anything to complete space

$E_{\bar{r}\bar{o}}$ = space such that $E_{\bar{o}} = E_{\bar{r}\bar{o}} \oplus E_{r\bar{o}}$

$E_{\bar{r}o}$ = —"— completes $\mathbb{R}^n$

Theorem: Given A, B, C, $\exists$ coordinates $z = Tx$ s.t. in z-coords

$\tilde{A} = \begin{bmatrix} A_{ro} & 0 & * & 0 \\ * & A_{r\bar{o}} & * & * \\ 0 & 0 & A_{\bar{r}o} & 0 \\ 0 & 0 & * & A_{\bar{r}\bar{o}} \end{bmatrix} \qquad \tilde{B} = \begin{bmatrix} B_{ro} \\ B_{r\bar{o}} \\ 0 \\ 0 \end{bmatrix} \qquad \tilde{C} = [C_{ro} \ \ 0 \ \ C_{\bar{r}o} \ \ 0]$

Balanced Representation:

Can show $\exists\, T$ s.t.

$\tilde{W}_c = \tilde{W}_o = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix}, \ z = Tx \qquad \sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n > 0$

$\tilde{W}_c \tilde{W}_o = \begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_n^2 \end{bmatrix}$ ← don't care about these

CDS 131

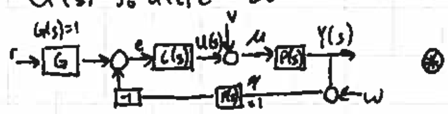
11/1 L10 cont'd:

Suppose that we drop the last k states in the balanced representation.

______ $\|G-\hat{G}\|_\infty \le 2\sum_{i=0}^{k-1}\sigma_{n-i}$ "twice the sum of tails"

11/4 L11: FREQUENCY DOMAIN

$U(s)=\int_0^\infty u(t)e^{-st}dt$



State space → Transfer functions

$Y(s)=C(sI-A)^{-1}x(0)+\underbrace{C(sI-A)^{-1}BU(s)}_{\text{TF: }H(s)}$

Initial Condition response

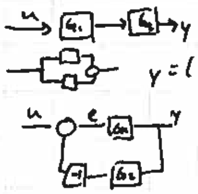
$H(s)=\frac{n(s)}{d(s)}=\frac{b_1 s^{n-1}+b_2 s^{n-2}+\dots+b_{n-1}s+b_n}{s^n+a_1 s^{n-1}+a_2 s^{n-2}+\dots+a_n}$ $n(s)=0 \Rightarrow$ zeros, $d(s)=0 \Rightarrow$ poles

$n(s)$: $b_1=CB$, $b_k=CA^{k-1}B+\sum_{i=1}^{k-1}a_i CA^{k-i-1}B$

$d(s)=\det(sI-A)$ "Markov parameters"

There are many state space "realizations" of transfer functions

Ex: $A=\begin{bmatrix}-a_1 & 1 & & \\ -a_2 & & \ddots & 0 \\ \vdots & & & 1 \\ -a_n & 0 & & 0\end{bmatrix}$ $B=\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}$ $C=[b_1\ b_2\ \dots\ b_n]$



$y=G_2G_1u$

$y=(G_1+G_2)u$

$e=u-G_2Y$; $Y=G_1e$ matrix of transfer functions

$y=(I+G_1G_2)^{-1}G_1u$

Properties: Stability, Performance, Robustness

Stability: I/O stability, BIBO stability

$H(s)$ is BIBO stable iff poles of $H(s)$ are in open LHP

Definition: A feedback system is internally stable if all I/O transfer functions are stable

$$\begin{bmatrix}e\\u\\\eta\end{bmatrix}=\begin{bmatrix}\frac{G}{1+PC} & \frac{-P}{1+PC} & \frac{-1}{1+PC}\\ \frac{CG}{1+PC} & \frac{1}{1+PC} & \frac{-C}{1+PC}\\ \frac{PCG}{1+PC} & \frac{P}{1+PC} & \frac{1}{1+PC}\end{bmatrix}\begin{bmatrix}r\\v\\w\end{bmatrix}$$ ⇝ assuming F=1

Theorem: ⊛ is internally stable iff $n_Pn_Cn_F+d_Pd_Cd_F$ has no RHP zeros (assuming $G(s)$ is stable)

Proof: (F=1) Sketch

$$\begin{bmatrix}e\\u\\\eta\end{bmatrix}=\frac{1}{n_Pn_C+d_Pd_C}\begin{bmatrix}d_Pd_C & -n_Pd_C & d_Pd_C\\ d_Pn_C & d_Pd_C & d_Pn_C\\ n_Pn_C & n_Pd_C & d_Pd_C\end{bmatrix}\begin{bmatrix}r\\v\\w\end{bmatrix}$$

If $(n_Pn_C+d_Pd_C)(s_0)=0$ for $Re(s_0)>0$, show that at least one of the xfer fcns will be unstable. Need to show that s_0 will not be in numerator for at least one of the 9 entries

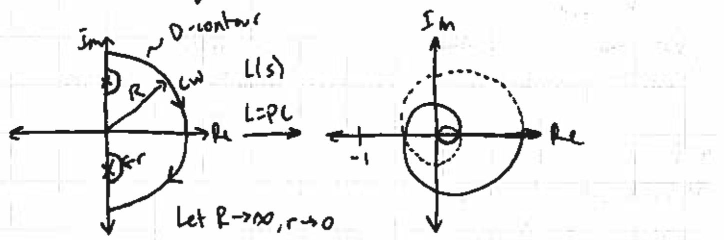
⊛ CDS QUALS QUESTION

Theorem 2: ⊛ is internally stable iff the following two conditions hold:

(1) 1+PCF has no RHP zeros

(2) No RHP pole/zero cancellations in PCF

Theorem 3: Nyquist



-1 is special because of the 1+PC condition.

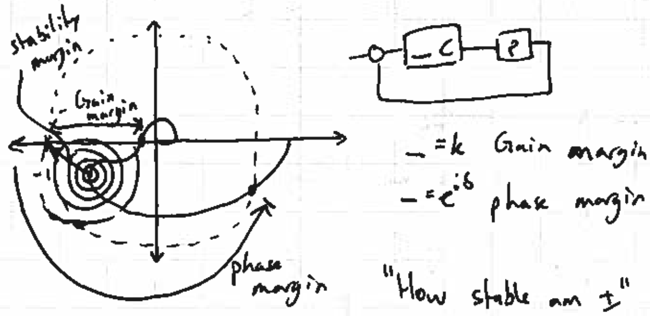
net CW encirclements = − # $\underline{\text{unstable poles of } P, C}$ + # $\underline{\text{unstable zeros of } 1+P}$

$N = -P + Z$

can count, given, how many unstable cl pole

MIMO: plot max singular value of $L(s)$

Intuition of Nyquist



_ = k Gain margin

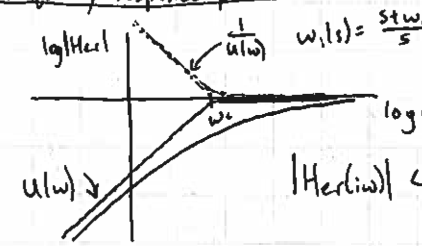
_ = $e^{i\delta}$ phase margin

"How stable am I"

Performance

Asymptotic tracking: Given r(t) find controller C such that (P,C) is internally stable & $\lim_{t\to\infty}(y(t)-r(t))=0$

Frequency response profiles:



$w_1(s)=\frac{s+w_c}{s}$

$|H_{er}(i\omega)|<U(\omega)$

Rewrite: $\|W_pH_{er}\|_\infty<1$ → Make W_p large in regions where you want small error: W_p "=" $\frac{1}{U(\omega)}$

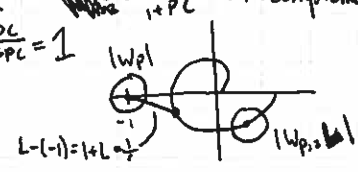
Choose $W_p(s)$ such that $|W_p(i\omega)|\simeq\frac{1}{U(\omega)}$

$H_{er}=\frac{1}{1+PC}=:S$ → Sensitivity function

$\|W_pS\|_\infty<1$ weight sensitivity

$\|W_{p_2}H_{\eta n}\|_\infty<1$ $H_{\eta n}=\frac{PC}{1+PC}=:T$ → complementary

$S+T=\frac{1}{1+PC}+\frac{PC}{1+PC}=1$

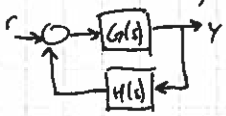


11/8 Recitation: Nyquist:

Time Domain	Frequency Domain
$G: \dot{x} = Ax + Bu$	$G(s)$
$y = Cx + Du$	Stability: $Re(poles(G(s))) < 0$
Stability: $Re(eig(A)) < 0$	$G(s) = C(sI - A)^{-1}B + D$
$det(\lambda I - A) = 0$	$poles(G(s))$
$roots(det(\lambda I - A))$	$roots(det(sI - A))$

Why use Nyquist?

1) Determine CL stability from OL information
2) Better understanding of structural aspects of stability
3) Geometric tool to visualize frequency response
4) Relative stability
5) Robust stability



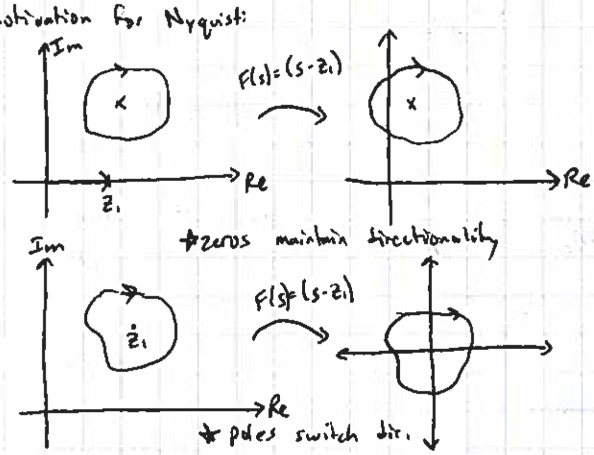
OLTF: GH
CLTF: $\frac{G}{1+GH}$

__Nyquist Criterion:__ $G(s) = \frac{N_G}{D_G}$ $H(s) = \frac{N_H}{D_H}$

$F(s) := $ denominator of CLTF $= 1 + GH = 1 + \frac{N_G N_H}{D_G D_H}$

$= \frac{D_G D_H + N_G N_H}{D_G D_H}$ $\rightarrow$ poles(CLTF) $\rightarrow$ poles(OLTF) (*)

Motivation for Nyquist:



If encircles the pole, it is shifted to origin

- Because OL = F(s) − 1, we map encirclements of poles to the value −1. This is why we care about encirclements of −1.
- Let ez = # encircled zeros, ep = # encircled poles
  - if ez = ep : No encirclements of origin
  - ez > ep : Same direction
  - ez < ep : opposite direction

→ zeros & poles of $F(s) - 1$

From (*), $\underbrace{N}_{\text{cw \# of encirclements of -1}} = \underbrace{Z}_{\text{\# CLTF RHP poles}} - \underbrace{P}_{\text{\# OLTF RHP poles}}$

$\therefore Z = N + P$ ← cw b/c zeros maintain direction ((cw) contour)

(*) Nyquist is just the complex plot of bode b/c Bode is $|L(j\omega)|$ for $\omega = 0 \rightarrow \infty$.

CDS 131 Exam

Ch 6:

- $h(t) = Ce^{At}B + D\delta(t)$
- $y(t) = Ce^{At}x_0 + \int_0^t h(t-\tau)u(\tau)d\tau$
- $e^{At} = I + At + \frac{1}{2}A^2t^2 + \dots = \sum_{k=0}^{\infty} \frac{1}{k!}A^k t^k$
- Consider the transformation to block Jordan form
  $\tilde{A} = TAT^{-1}$, $\tilde{B} = TB$, $\tilde{C} = CT^{-1}$
  Then because $e^{TAT^{-1}} = Te^{A}T^{-1}$, we have
  $x(t) = T^{-1}e^{\tilde{A}t}Tx(0) + T^{-1}\int_0^t e^{\tilde{A}(t-\tau)}\tilde{B}u(\tau)d\tau$
- $y(t) = \underbrace{CA^{-1}e^{At}B}_{\text{transient}} + \underbrace{D - CA^{-1}B}_{\text{steady-state}}$

Ch 7:

$W_c(t) = \int_0^T e^{A\tau}BB^Te^{A^T\tau}d\tau$

$0 \to x_f$: $u(t) = B^Te^{A^T(T-t)}W_c^{-1}(T)x_f$

- Reachability Tests:
  1) $W_c(T) \succ 0$ for any $T > 0$
  2) W_c satisfies $AW_c + W_c^TA^T = -BB^T$ and $W_c \succ 0$
     (and $W_c = \lim_{T\to\infty} W_c(T)$ for A stable)
  3) rank $[sI - A | B] = n$ $\forall s \in \mathbb{C}$
  4) rank $[B | AB | A^2B | \dots | A^{n-1}B] = n$
- Reachable canonical form:

$$\tilde{A} = TAT^{-1} = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & 1 & 0 \end{bmatrix} \quad \tilde{B} = T^{-1}B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$CT = [b_1 \ b_2 \ \cdots \ b_n]$ $\quad T = \tilde{W}_r W_r^{-1}$; $\tilde{W}_r \to [\tilde{B}\ \tilde{A}\tilde{B} \dots]$; $W_r^{-1} \to [B\ AB \dots]^{-1}$

- Stabilizeability:
  Pole placement of $(A - BK)$
- LQR: $J(x_0) = \int_0^{t_f}(x^TQ_xx + u^TQ_uu)dt + x^T(t_f)Q_fx(t_f)$
  Algebraic RE: $0 = PA + A^TP - PBQ_u^{-1}B^TP + Q_x$
  ↳ $u = -Q_u^{-1}B^TPx$ is optimal
  $Q_x \succeq 0$; $Q_u \succ 0$ to guarantee existence of solution
  ↳ $Q_u \succ 0$ because it needs to be invertible. (for sol.)
  ↳ $Q_x \succ 0$ so integral cost is zero iff $x = 0$
  ↳ can allow $\succeq 0$ if we don't care about some states

Ch 8:

$W_o(T) = \int_0^T e^{A^T\tau}C^TCe^{A\tau}d\tau$

$x_0 = (W_o(T))^{-1}\int_0^T e^{A^T\tau}C^Ty(\tau)d\tau$

- Observability tests:
  1) $W_o(T) \succ 0$ for any $T > 0$
  2) If A is stable, solution of $A^TW_o + W_oA = -C^TC$ is positive definite
  3) PBH: rank $\begin{bmatrix} C \\ sI - A \end{bmatrix} = n$ $\forall s \in \mathbb{C}$
  4) rank $\begin{bmatrix} C \\ CA \\ \vdots \end{bmatrix} = n$

- Observable canonical form:

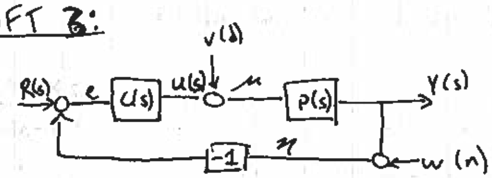
$$\frac{dz}{dt} = \begin{bmatrix} -a_1 & 1 & & 0 \\ -a_2 & 0 & \ddots & \\ \vdots & & \ddots & 1 \\ -a_n & & & 0 \end{bmatrix} z + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} u$$

$T = \tilde{W}_o^{-1}W_o$

$y = [1 \ 0 \cdots 0]z + d_0u$

- Observer: $\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$
  $\tilde{x} = x - \hat{x}$; $\dot{\tilde{x}} = (A - LC)\tilde{x}$
- Observability: Pole placement of $(A - LC)$

DFT 3:



$Y(s) = C(sI - A)^{-1}x_0 + \underbrace{C(sI - A)^{-1}B}U(s)$

$\frac{b_1s^{n-1} + \dots + b_{n-1}s + b_n}{s^n + a_1s^{n-1} + \dots + a_n} = H(s) = \frac{n(s)}{d(s)} = \frac{\text{zeros} \to n(s)=0}{\text{poles} \to d(s)=0}$

- Internally stable if all I/O tf are stable

$$\begin{bmatrix} e \\ u \\ n \end{bmatrix} = \frac{1}{1+PC}\begin{bmatrix} 1 & -P & -1 \\ C & 1 & -C \\ PC & P & 1 \end{bmatrix}\begin{bmatrix} r \\ v \\ w \end{bmatrix} \qquad \begin{bmatrix} e \\ u \end{bmatrix} = -\begin{bmatrix} PS & S \\ T & CS \end{bmatrix}\begin{bmatrix} d \\ n \end{bmatrix}$$

- Internally stable iff:
  a) $1 + PC$ has no zeros in $\text{Re } s \geq 0$
  b) PC has no pole/zero cancellations in $\text{Re } s \geq 0$.
- Nyquist:
  # net CW encirclements $= -$# unstable poles of PC + # unstable zeros of $1 + PC$

DFT 4: • $S = \frac{1}{1+PC} = H_{er}$; $T = \frac{PC}{1+PC} = H_{yn}$

- Robust Performance: $W_p \equiv W_1$; $W_r \equiv W_2$
- Robust stability: (Pg 56 DFT)

Perturbation	Condition	TYPE
$(1 + \Delta W_2)P$	$\|W_2T\|_\infty < 1$	Multiplicative
$P + \Delta W_2$	$\|W_2CS\|_\infty < 1$	Additive
$P/(1 + \Delta W_2P)$	$\|W_2PS\|_\infty < 1$	Feedback
$P/(1 + \Delta W_2)$	$\|W_2S\|_\infty < 1$	

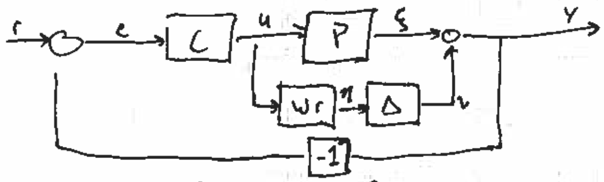
- Nominal performance condition $\|W_1S\|_\infty < 1$
- Robust Performance:

	Nominal Perf. Condition	
Perturbation	$\|W_1S\|_\infty < 1$	$\|W_1T\|_\infty < 1$
$(1 + \Delta W_2)P$	$\| \lvert W_1S\rvert + \lvert W_2T\rvert \|_\infty < 1$	~
$P + W_2\Delta$	$\| \lvert W_1S\rvert + \lvert W_2CS\rvert \|_\infty < 1$	~
$P/1 + \Delta W_2P$	~	$\| \lvert W_1T\rvert + \lvert W_2PS\rvert \|_\infty <$
$P/(1 + \Delta W_2)$	~	$\| \lvert W_1T\rvert + \lvert W_2S\rvert \|_\infty < 1$.

↑ plug & chug to get ↗ (63 DFT)

Small Gain Theorem: If $\|G\|_\infty < 1$ & $\|H\|_\infty \le 1$ and G & H are stable then the closed loop system is stable.

Ex:



$\eta = W_r u$; $u = Ce$; $e = v - Pu$

$\Rightarrow u = Ce = C(v - Pu) \Rightarrow u(PC+1) = Cv \to u = \frac{C}{1+PC} v$

$\to \eta = W_r C S v$

$\therefore H = W_r C S.$

← How to get robust stability condition.

DFT 5:

If $P \in S$, $\mathcal{C} = \left\{ \frac{Q}{1-PQ} : Q \in S \right\}$. If $P \notin S$, $\mathcal{C} = \left\{ \frac{X+MQ}{Y-NQ} : Q \in S \right\}$

Theorem 3: P is strongly stabilizable iff it has an even # of real poles between every pair of real zeros in $\mathrm{Re}\, s \ge 0$.

DFT 6: Limits of performance

-check first for stability

Algebraic Limits: (always true)

- $S + T = 1$
- $\min\{|W_1(j\omega)|, |W_2(j\omega)|\} < 1 \quad \forall \omega$
- If p is a pole of L in $\mathrm{Re}\, s \ge 0$ & z is zero in $\mathrm{Re}\, s \ge 0$:

$$S(p) = 0 \quad S(z) = 1 \qquad \Longrightarrow \|W_1 S\|_\infty \ge |W_1(z)|$$
$$T(p) = 1 \quad T(z) = 0 \qquad \|W_2 T\|_\infty \ge |W_2(p)|$$

Maximum modulus principle: if stable & bounded (analytic), max will occur on $j\omega$ axis.

Bode's Integral formula:

$$\int_0^\infty \ln|S(i\omega)|\, d\omega = \pi \sum_{p_i \in \text{RHP (of } L)} p_i$$

All-pass TF:

All pass: $H(s) \rightsquigarrow |H(i\omega)| = 1 \quad \forall \omega.$

Theorem: $G(s) = G_{mp} \cdot G_{ap}$

no RHP poles/zeros ↑ ↑ All pass

Ex (14.9 FBS pg 19)

$P(s) = \frac{n(s)}{d(s)} \tilde{P}(s)$

$$S(s) = \frac{1}{1+P(s)C(s)} = \frac{d(s)}{d(s) + n(s)\tilde{P}(s)C(s)} \; ; \; W_p = \frac{d(-s)}{d(s)}$$

$$M_S = \max_\omega |S(i\omega)| = \max |W_p(i\omega) S(i\omega)| \ge |W_p(z_k) S(z_k)| = \left| \frac{d(-z_k)}{d(z_k)} \right|$$

The waterbed effect:

Suppose P has a zero w/ $\mathrm{Re}\, z > 0$. Then $\exists\, c_1, c_2$ as functions only on ω_1, ω_2, z such that

$$c_1 \log M_1 + c_2 \log M_2 \ge \log|S_{ap}(z)^{-1}| \ge 0.$$

where $M_1 := \max_{\omega_1 \le \omega \le \omega_2} |S(j\omega)|$; $M_2 := \|S\|_\infty$

CDS 131

11/11 L 13: Uncertainty & Robustness:

$$P\begin{cases}\dot{x} = f(x,u,\boxed{d,\theta}\,z) & \theta\in\Theta\\ y = h(x,\theta,\boxed{n})\ \text{uncertainty}\end{cases}$$

Q: When is the CL system stable $\forall\,\theta\in\Theta$

Q: When is performance satisfied — // —

$\dot{z} = g(x,z,\theta)$ unmodeled dynamics $g\in\mathcal{G}$

→ In addition, $\forall\, g\in\mathcal{G}$

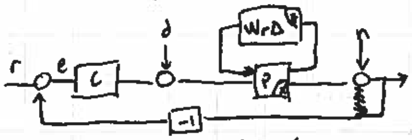
n,d exogenous inputs not affected by x,z

z coupled dynamical uncertainty

θ often constant approximation for dynamic quantity

$\dot{\theta}=0$ is a special case of $\dot{z}=\dots$

Linear (Control) Systems:



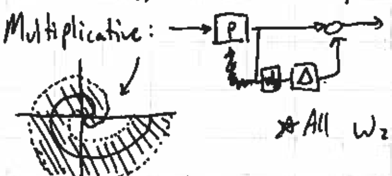
Processes: $P\in\mathcal{P}=\{\tilde{P}:\|\Delta\|_\infty<1\}$

Robust stability $\Leftrightarrow$ internally stable $\forall P\in\mathcal{P}$

Performance $\Leftrightarrow$ $\|W_p S\|_\infty\le 1$ $\forall P\in\mathcal{P}$

Common types of uncertainty: $\tilde{L}=\tilde{P}C$

Multiplicative:

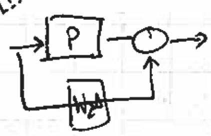


$\tilde{P}=P(1+W_2\Delta)$

$\|\Delta\|_\infty<1$

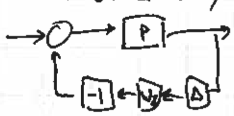
*All W_2's are W_r's

Additive:



$\tilde{P}=P+W_2\Delta$

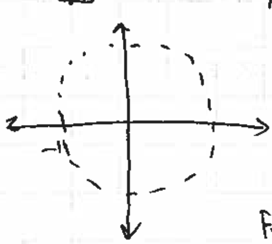
Feedback Uncertainty:

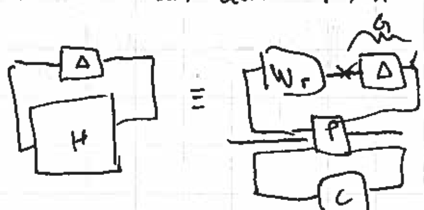


$\tilde{P}=\frac{P}{1+P\Delta W_2}$

Robust Stability:

Thm (small gain) If $\|G\|_\infty<1$ & $\|H\|_\infty\le 1$ and G, H are stable then closed loop is stable





Feedback: $H=-PW_r$

Multiplicative: $H=\frac{W_rPC}{1+PC}\rightarrow\|W_rT\|_\infty<1$

complementary sensitivity func $T=\frac{PC}{1+PC}=1-S$

More generally: Assume W_r stable

Theorem: $\hat{P}=P(1+W_r\Delta)$, $\|\Delta\|_\infty<1$ is internally stable $\Leftrightarrow\|W_rT\|_\infty<1$

Proof: (unstable open loop): key: look for changes of encirclements of -1

$$1+\tilde{P}C=1+PC(1+W_r\Delta)$$
$$=(1+PC)\left(1+\frac{PC}{1+PC}W_r\Delta\right)$$

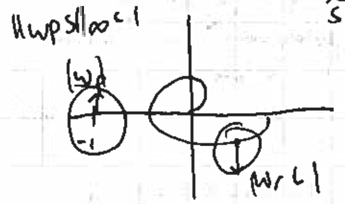
$\|TW_r\Delta\|_\infty\le\|W_rT\|_\infty\|\Delta\|_\infty<1$

Robust Performance:

Performance criterion: $\|W_pS\|_\infty\le 1$

Robust performance $\Leftrightarrow$ robust stability & $\|W_p\hat{S}\|_\infty<1$

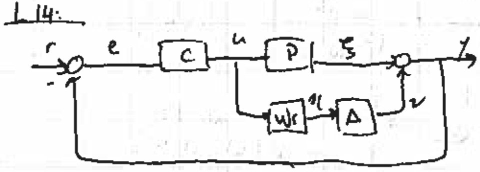
$\hat{S}=\frac{1}{1+\hat{P}C}$ $\hat{P}\in\dots$

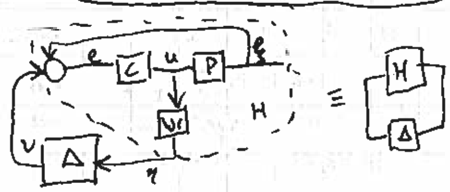


$$\|\,|W_pS|+|W_rT|\,\|_\infty<1$$

11/13 L 14:

Ex:





$\eta=W_ru$

$u=Ce$

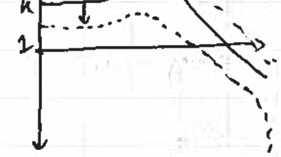
$e=v-Pu$

$\rightarrow u=Ce=C(v-Pu)\rightarrow u(1+PC)=Cv\rightarrow u=\frac{C}{1+PC}v$

$\eta=W_rCSv$

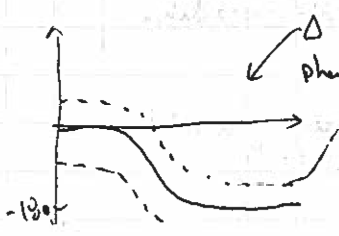
Ex: $P=\frac{k}{s^2+2\zeta\omega_0 s+\omega_0^2}$ $k=k_{nom}\pm 50\%$

$\tilde{P}=P(1+0.5\Delta)$, $\|\Delta\|_\infty\le 1$, $W_r=0.5$

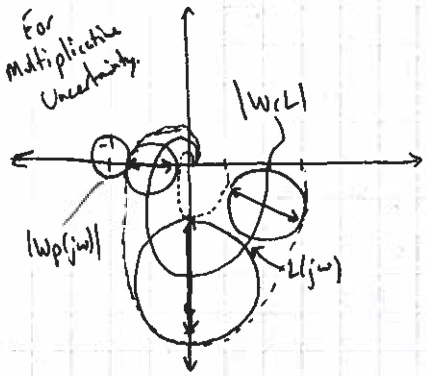


can potentially diverge at high frequencies if you have low certainty

Δ allows for phase changes



4/13 L14: cont'd:



Robust Performance:

Theorem 2: If P has multiplicative uncertainty with weight W_r and performance spec is $\|W_p S\|_\infty < 1$ then the system has robust performance ($\|W_p \tilde{S}\|_\infty < 1$, $\tilde{S} = \frac{1}{1+\tilde{P}C}$, $\tilde{P} \in \mathcal{P}_{mult}(W_r)$) iff $\| |W_p S| + |W_r T| \|_\infty < 1$

nominal performance ↗ robust ↑ stability

Aside: $\sup_{u \in L_2} \frac{\|y\|_2}{\|u\|_2} = \|G\|_\infty$ iff G is stable!

∴ Roughly, we need to show (1) robust stability & (2) $\|W_p \tilde{S}\|_\infty < 1$

$$\|W_p \tilde{S}\|_\infty = \left\| \frac{W_p}{1+\tilde{P}C} \right\|_\infty = \left\| \frac{W_p}{1+P(1+W_r\Delta)C} \right\|_\infty \quad \forall \Delta$$

$$\frac{1}{1+PC(1+W_r\Delta)} = \frac{1}{1+PC} \cdot \frac{(1+PC)}{(1+PC)+PC\Delta W_r} = S \frac{1}{1+W_r T \Delta}$$

$$1 > \left| \frac{W_p S}{1-|W_r T|} \right| \geq \left| \frac{W_p S}{1+\Delta W_r T} \right| \quad \forall |\Delta| \leq 1 \text{ assuming } |W_r T| < 1, \ \forall s = iw$$

↑ show this.

$$\frac{|W_p S|}{1-|W_r T|} < 1 \rightarrow |W_p S| < 1 - |W_r T|$$

$$|W_r T| + |W_p S| < 1 \quad \forall s \in jw$$

Perturbation	Stability	$\|W_p S\| < 1$	$\|W_r t\| < 1$
$P(1+W_r\Delta)$	$\|W_r T\|_\infty < 1$	$\|W_r T + W_p S\|_\infty < 1$	messy...
$P + W_r\Delta$	$\|W_r C S\|_\infty < 1$	$\|W_p S + W_r C S\|_\infty < 1$	messy...
$\cancel{P}(1+W_r\Delta)$	$\|W_r S\|_\infty < 1$	messy...	$\|W_r S + W_p T\|_\infty < 1$

Recitation:

$P_1 = \frac{1}{s+1} \qquad P_2 = \frac{1}{s+a}$

Show: P_1 can be turned into P_2 w/ bounded additive & multiplicative uncertainty if $a > 0$ not if $a < 0$. Also, no restriction on a if feedback uncertainty is used.

Strategy: P_1 = nominal system (P)
P_2 = perturbed system ($\tilde{P}$)

Find allowable Δ, W_z s.t. by tuning Δ, $P_1 \to P_2$

Additive:

$$\tilde{P} = \frac{1}{s+a} = P_1 + \Delta_a W_a = \frac{1}{s+1} + \Delta_a W_a$$

$$\Delta_a W_a = \frac{1-a}{(s+1)(s+a)} \rightsquigarrow d := 1-a \rightsquigarrow \Delta_a W_a = \frac{d}{(s+1)(s+1-d)}$$

$\to P_1: d = 0,\ P_2: d = 1-a$

$\Delta_a W_a$ bounded $\Leftrightarrow \frac{d}{(s+1)(s+1-d)}$ no RHP $\Leftrightarrow 1-d > 0 \Leftrightarrow a > 0$

Multiplicative:

$$(1+\Delta_m W_m)P_1 = P_2 \rightsquigarrow \Delta_m W_m = \frac{1-a}{s+a}$$

Feedback:

$$P_2 = P_1 / (1+\Delta_f W_f P_1) \rightsquigarrow \Delta_f W_f = a-1$$

$\Delta_f W_f$ bounded $\Leftrightarrow$ no restriction on a.

*Exercise: $P_1 = \frac{s+1}{(s+1)^2} \qquad P_2 = \frac{s+a}{(s+1)^2}$

Show: add + Mult. no restriction on a
feedback: $a > 0$.

Theorem 2: Robust Performance $\Longleftrightarrow \| |W_1 S| + |W_2 T| \|_\infty < 1$

$\rightsquigarrow$ Robust performance $\begin{cases} \|W_2 T\|_\infty < 1 \\ \left\|W_1 \frac{S}{1+\Delta W_2 T}\right\|_\infty \leq 1 \end{cases}$ $\forall$ allowable Δ

WTS $\Leftrightarrow \| |W_1 S| + |W_2 T| \|_\infty < 1$.

Proof: $\| |W_1 S| + |W_2 T| \|_\infty < 1$

↳ $s = jw$ $\begin{cases} |W_2 T| < 1 \\ |W_1 S| + |W_2 T| < 1 \end{cases}$ $\forall w$

↳ $\begin{cases} |W_2 T| < 1 \\ |W_1 S| / 1 - |W_2 T| < 1 \end{cases}$ $\forall w$

↳ $\begin{cases} \|W_2 T\|_\infty < 1 \\ \|W_1 S / 1 - |W_2 T|\|_\infty < 1 \end{cases}$

$$\max_{|\Delta| < 1} \left\| \frac{W_1 S}{1+\Delta W_2 T} \right\|_\infty = \left\| \frac{W_1 S}{1+|W_2 T|} \right\|_\infty$$

$$\Leftrightarrow \max_{|\Delta| < 1} \max_{w_1} \left| \frac{W_1 S}{1+\Delta W_2 T} \right| = \max_{w_2} \left| \frac{W_1 S}{1-|W_2 T|} \right|$$

can switch these (LHS) (RHS)

2. Pick w_1 s.t. max is achieved
$\exists \Delta$ s.t. $|1+\Delta W_2 T| = 1 - |W_2 T|$ at w_2

$$\Leftrightarrow \max_{|\Delta| < 1} \left| \frac{W_1 S}{1+\Delta W_2 T} \right| \geq \left| \frac{W_1 S}{1-|W_2 T|} \right| \rightsquigarrow \left\| \frac{W_1 S}{1-|W_2 T|} \right\| \quad \text{RHS}$$

$$↳ \leq \max_{w_1} \max_{|\Delta| < 1} \left| \frac{W_1 S}{1+\Delta W_2 T} \right| \quad \text{(LHS)}$$

$\leq$. $1 = |1+\Delta W_2 T - \Delta W_2 T|$

$1 \leq |1+\Delta W_2 T| + |\Delta W_2 T|$, $\leq |\Delta||W_2 T| \leq |W_2 T|$ if allowable

$\to |1+\Delta W_2 T| \geq 1 - |W_2 T| \quad \forall \Delta$

$$\frac{1}{|1+\Delta W_2 T|} \leq \frac{1}{1-|W_2 T|} \rightarrow \frac{|W_1 S|}{|1+\Delta W_2 T|} \leq \frac{|W_1 S|}{1-|W_2 T|} \quad \forall w$$

$$\left\| \frac{W_1 S}{1+\Delta W_2 T} \right\|_\infty \leq \left\| \frac{W_1 S}{1-|W_2 T|} \right\|_\infty \rightarrow \text{LHS} \leq \text{RHS} \quad \therefore = \blacksquare$$

11/18: Other types of questions:

1. Given P, what is the set of all controllers $\mathcal{C}$ that stabilize (P, C), $C \in \mathcal{C}$?
2. Given controller C that stabilizes some process P, what is the set of all P that C stabilizes.
3. Given P, what are the fundamental limits of performance for any controller C?

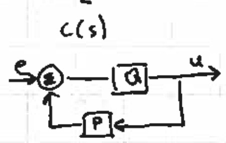
1. Let $S = \{P : P \text{ stable \& proper}\}$

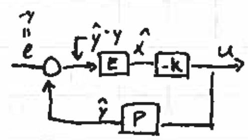
Case 1: If P is stable, then

$$C \text{ stabilizing} \iff C = \frac{Q}{1+PQ} \quad Q \in S$$

Proof: $Q = \frac{C}{1+PC} = CS$

$$\begin{bmatrix} S & CS \\ PS & T \end{bmatrix} = \begin{bmatrix} 1-PQ & Q \\ P(1-PQ) & PQ \end{bmatrix}$$





Case 2: P is unstable: need coprime factorization

$$P = N_p M_p^{-1} \text{ (right coprime factorization)}$$
$$= \tilde{M}_p^{-1} \tilde{N}_p \text{ (left coprime factorization)}$$

$M_p, N_p \in S$ with no common factors (p/z cancellation)

Easy coprime factorization

$$N_p = \frac{n(s)}{(s+\lambda)^n} \; ; \; M_p = \frac{d(s)}{(s+\lambda)^n} \qquad \frac{N_p}{M_p} = \frac{n}{d}$$

Theorem: $N, M \in S$ coprime
$\Updownarrow$
$\exists X, Y \in S$ s.t. $MX + NY = I$

↳ intuition: generalization of Bezout's identity.

Theorem: C stabilizing $\iff C = (X+MQ)/(Y-NQ)$ $Q \in S$

⇝ Youla-parameterization (Q-parameterization)

$S = M(Y - NQ)$ affine in Q

$T = N(X + MQ)$

Example:

$$P = \frac{s^2+2s+1}{s^3+1} \begin{matrix} \}a \\ \}b \end{matrix}$$

$$\underbrace{s^2+2s+1}_{a} = \underbrace{(s^3+1)}_{b}\underbrace{(0)}_{q} + \underbrace{(s^2+2s+1)}_{r}$$

$a = bq + r$, $\gcd(a,b) = \gcd(b,r) \ldots = r \neq 0$

Starting over:

$s^2+2s+1 = (s^3+1)(0) + (s^2+2s+1)$

$(s^3+1) = (s^2+2s+1)(s-2) + (3s+3)$ → last nonzero remainder, ∴ = gcd.

$s^2+2s+1 = (3s+3)\left(\frac{s}{3}+\frac{1}{3}\right) + (0)$

↳ $(s+1) = (s^2+2s+1)\left(\frac{2-s}{3}\right) + \left(\frac{s^3+1}{3}\right)$

$1 = \frac{s^2+2s+1}{s+1}\left(\frac{2-s}{3}\right) + \frac{s^3+1}{3(s+1)}$

↳ Will not give us $Q \in S$., we need something else

↳ Procedure B (DFT Ch. 5).

Map: $s = \frac{1-\lambda}{\lambda}$, $\lambda = \frac{1}{(s+1)}$; $\tilde{P}(\lambda) = \frac{n(\lambda)}{m(\lambda)}$ → GCD again

→ Do above again and it works out

$$P = \frac{s^2-s-2}{s^3-2s^2-3s}, \quad s = \frac{1-\lambda}{\lambda}$$

$$\tilde{P} = \frac{\left(\frac{1-\lambda}{\lambda}\right)^2 - \left(\frac{1-\lambda}{\lambda}\right) - 2}{\left(\frac{1-\lambda}{\lambda}\right)^3 - 2\left(\frac{1-\lambda}{\lambda}\right)^2 - 3\left(\frac{1-\lambda}{\lambda}\right)} = \frac{\lambda - 4\lambda^2}{-6\lambda^3+4\lambda^2+\lambda+1} \begin{matrix} \}a \\ \}b \end{matrix}$$

Long Div: $(-6\lambda^3+4\lambda^2+\lambda+1) = (-4\lambda^2+\lambda)\left(\frac{3\lambda}{2} - \frac{5}{8}\right) + \left(\frac{13\lambda}{8}+1\right)$

$(-4\lambda^2+\lambda) = \left(\frac{13\lambda}{8}+1\right)\left(-\frac{32\lambda}{13} + \frac{360}{169}\right) + \left(-\frac{360}{169}\right) = r$, next step $= 0$.

↳ $1 = \underbrace{\left(\frac{169}{360} + \frac{3\lambda}{2} - \frac{5}{8}\right)}_{X\,(Y)}\underbrace{(4\lambda^2 - \lambda)}_{M\,(N)} + \left(\frac{169}{360}\right)\underbrace{(-6\lambda^3+4\lambda^2+\lambda+1)}_{N\,(M)}\underbrace{\left(-\frac{32\lambda}{13} + \frac{36}{16}\right.}_{Y\,(X}$

→ convert back to s via $\lambda = \frac{1}{s+1}$, $P = NM^{-1}$

$C = \frac{X+MQ}{Y-NQ}$, $Q \in S$. this controller will stabilize P.

State space form:

$$P(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = C(sI-A)^{-1}B + D$$

$$\begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \qquad \begin{matrix} \dot{x} = Ax + Bu \\ y = Cx + Du \end{matrix}$$

Have k: $(A - Bk)$ is stable.

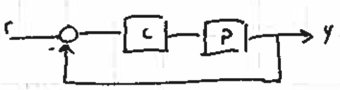
Mapping $v = u + kx$; $u = v - kx$

$\dot{x} = (A - Bk)x + Bv$; $y = (C - Dk)x + Dv$; $u = v - kx$

$P = \frac{Y(s)}{U(s)} = \frac{Y(s)}{V(s)} \cdot \frac{V(s)}{U(s)} = NM^{-1}$

$$\underset{u \text{ to } y}{N} = \left[\begin{array}{c|c} A - Bk & B \\ \hline C - Dk & D \end{array}\right] \qquad \underset{v \text{ to } u}{M} = \left[\begin{array}{c|c} A - Bk & B \\ \hline -k & I \end{array}\right]$$

11/20 Recitation: Coprime Factorization.



Recap: S: family of all stable, proper, real, rational functions

Case 1: If P is stable: $\exists\, C$ stabilizing

$$\iff C = \frac{Q}{1+PQ}, \quad Q \in S$$

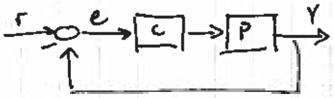
Case 2: If P is unstable: $\exists\, C$ stabilizing

$$\iff C = \frac{X+MQ}{Y-NQ}, \quad Q \in S$$

Right $\left\{ \begin{array}{l} P = NM^{-1} \\ MX + NY = I \end{array} \right\}$ (Youla Parameterization)

Left. $\left\{ \begin{array}{l} P = M^{-1}N \\ MX + NY = I \end{array} \right\}$

11/22:



$S = \{H : H$ is stable, proper$\}$

$P = NM^{-1}$ $N, M \in S$ and coprime

Coprime: $NX + MY = I$, $X, Y \in S$

(No shared RHP zeros for N, M)

Youla Parameterization:

C stabilizing $\iff C = \frac{X+MQ}{Y-NQ}$, $Q \in S$

$S = \frac{1}{1+PC} = M(Y-NQ)$

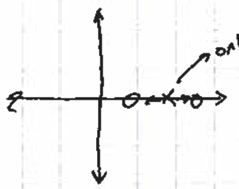
$T = \frac{PC}{1+PC} = N(X+MQ)$

(If P stable, choose $N = P, M = 1, X = 0, Y = 1$)

Strong stabilization:

P is strongly stabilizable if $\exists$ $C \in S$ that stabilizes the system.

Intuition: Root locus



Theorem: P is strongly stabilizable $\iff$ even # of unstable real poles between every pair of "unstable" real zeros (including $s = +\infty$)

Proof: $P = N/M$ $NX + MY = 1$, $M, N, X, Y \in S$

$C = (X+MQ)/(Y-NQ)$ $Q \in S$

Suppose P has zeros at $\sigma_1, \sigma_2 > 0$, $\sigma_1, \sigma_2 \in \mathbb{R}$

Then $N(\sigma_1) = N(\sigma_2) = 0$ (because $M \in S$)

Suppose have odd # of poles of P b/w σ_1/σ_2

Claim that $M(\sigma_1)$ & $M(\sigma_2)$ have opposite sign

→ sign of M as I go through a zero of M (pole of P) changes from +1 to -1 or opposite.

$M(\sigma_1)Y(\sigma_1) = 1 = M(\sigma_2)Y(\sigma_2)$

$\Rightarrow Y(\sigma_1)$ & $Y(\sigma_2)$ have opposite sign.

$C = \frac{X+MQ}{Y-NQ}$ $Y-NQ$ also changes sign between σ_1 & σ_2

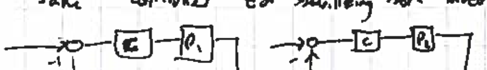
$\Rightarrow Y-NQ$ has at least 1 zero b/w σ_1 & σ_2

$\Rightarrow C$ has a pole b/w σ_1 & σ_2 for any Q

Other direction is harder (look in DFT)

Q: When can a single controller C stabilize two processes P_1 & P_2.

(~ I am taking off and landing, can I use the same controller for stabilizing both modes?)



→ Nowadays would just change the code

Theorem: $P_1 = N_1/M_1$, $P_2 = N_2/M_2$ are simultaneously stabilizable iff $P = (N_2M_1 - N_1M_2)/(N_2X_1 + M_2Y_1) = N/M$ is strongly stabilizable

Proof: Find Q_1 & Q_2 s.t. $C_1 = \frac{X_1+M_1Q_1}{Y_1-N_1Q_1} = \frac{X_2+M_2Q_2}{Y_2-N_2Q_2} = C_2 \iff \exists u \in S$ s.t.

$\begin{cases} X_1 + M_1Q_1 = u(X_2+M_2Q_2) \\ Y_1 - N_1Q_1 = u(Y_2 - N_2Q_2) \end{cases}$

$\xrightarrow{\text{algebra}} M + NQ_1 = u \iff MM_c + NN_c \in S$, $C = N_c/M_c$

Q: Given (P, C) stable, what is a set $\mathbb{P}$ such that (P_1, C) is stable $\forall P_1 \in \mathbb{P}$.

Chordal Distance:

$$d_{P_1P_2}(\omega) = \frac{|P_1(j\omega) - P_2(j\omega)|}{\sqrt{1+|P_1(j\omega)|^2}\sqrt{1+|P_2(j\omega)|^2}}$$

ν-gap metric: $\delta_\nu(P_1, P_2) = \begin{cases} \sup_\omega d_{P_1P_2}(\omega) & \text{if } 1+P_1(j\omega)P_2(j\omega) \neq 0 \text{ \& winding \# condition} \\ 1 & \text{o.w.} \end{cases}$

Generalized stability margin:

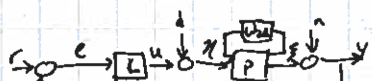
$\sigma_m = \begin{cases} \inf_\omega d_{P_1, -1/C}(\omega) & \text{if } C \text{ stabilizes} \\ 0 & \text{o.w.} \end{cases}$

Why $P_1, -1/C$?

If $d_{P_1,-1/C}(\omega) = 0$ $P(j\omega) = \frac{-1}{C(j\omega)}$; $1 + P(j\omega)C(j\omega) = 0$

Theorem: C stabilizes all P_1 s.t. $\delta_\nu(P_1, P) < \sigma_m(P, C)$ and "essentially" $d_{P_1P_2}(\omega) < d_{P_1,-1/C}(\omega)$ $\forall \omega$

11/25 Limits on performance : DFT ch. 6



Algebraic limits:

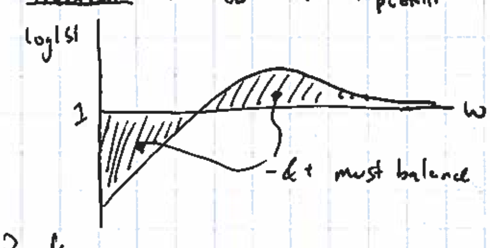
$S + T = 1$; $\| |W_pS| + |W_rT| \|_\infty < 1$

$\min(|W_p(j\omega)|, |W_r(j\omega)|) < 1$ $\forall \omega$

★ Bode's Integral Formulation:

Assume rel deg $L = PC \geq 2$ & internally stable

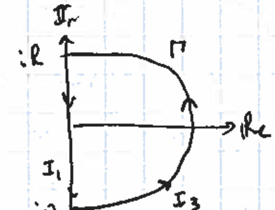
Theorem: $\int_0^\infty \log|S(i\omega)| d\omega = \pi \sum_{p_i \in RHP} p_i$ ← poles of L



Water-bed effect.

Proof:

$\int_\Gamma \log S(s) ds = \lim_{R\to\infty} I_1 + I_3$



$I_1 = \int_R^{-R} \log S(i\omega) d(i\omega)$

$= -2i \int_0^R \log|S(i\omega)| d\omega$

$I_3 = R \int_{-\pi/2}^{\pi/2} \log(1 + L(Re^{i\theta})) d\theta$

$= 0$ if L has rel deg ≥ 2.

$\log Me^{i\theta} = \log M + i\theta$

$S = \frac{1}{1+L}$

$\log S = -\log(1+L)$

12/4 Cont'd: Proof of Lemma 3 in DFT

Lemma 3: $\forall s_0 = \sigma_0 + j\omega_0$ s.t. $\sigma_0 > 0$

$$\log|S_{mp}(s_0)| = \frac{1}{\pi}\int_{-\infty}^{\infty} \log|S(j\omega)| \frac{\sigma_0}{\sigma_0^2 + (\omega - \omega_0)^2} d\omega$$

Proof: Use Lemma 1: $F(s_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} F(j\omega) \frac{\sigma_0}{\sigma_0^2+(\omega-\omega_0)^2} d\omega$

Plug in $F(s) = \ln S_{mp}(s)$, take real part.

- $Re[F(s)] = \frac{1}{\pi}\int_{-\infty}^{\infty} Re[F(j\omega)] \frac{\sigma_0}{\sigma_0^2+(\omega-\omega_0)^2} d\omega$
- $S_{mp}(s) = e^{F(s)} = e^{Re[F(s)]} e^{jIm[F(s)]}$

  $|S_{mp}(s)| = e^{Re[F(s)]} \Rightarrow Re[F(s)] = \ln|S_{mp}(s)|$ (LHS)
- Since $S = S_{mp}(s)\, S_{ap}(s)$ with $|S_{ap}(s)| = 1$

  $\Rightarrow |S(j\omega)| = |S_{mp}(j\omega)|$

$$\ln|S_{mp}(s_0)| = \frac{1}{\pi}\int_{-\infty}^{\infty} \ln|S(j\omega)| \frac{\sigma_0}{\sigma_0^2+(\omega-\omega_0)^2} d\omega.$$

Theorem 1 (Waterbed):

Suppose P has a zero at z in RHP. Then $\exists c_1, c_2$ constants of ω_1, ω_2, z where $M_1 := \max_{\omega_1 \le \omega \le \omega_2} |S(j\omega)|$

$M_2 := \|S\|_\infty$

$$c_1 \log M_1 + c_2 \log M_2 \ge \log|S_{ap}(z)^{-1}| \ge 0$$

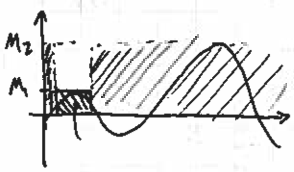
Proof:

$S(z) = 1 = S_{ap}(z) S_{mp}(z) \Rightarrow S_{mp}(z) = \frac{1}{S_{ap}(z)}$

$\log|S_{ap}(z)^{-1}| = \frac{1}{\pi}\int_{-\infty}^{\infty} \log|S(j\omega)| \frac{\sigma_0}{\sigma_0^2+(\omega-\omega_0)^2} d\omega$

$c_1 = \frac{1}{\pi}\int_{I_1} \frac{\sigma_0}{\sigma_0^2+(\omega-\omega_0)^2} d\omega \quad I_1 = [-\omega_2, \omega_1] \cup [\omega_1, \omega_2]$

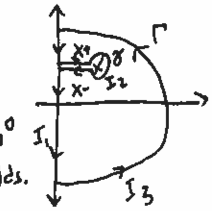
$c_2 = \frac{1}{\pi}\int_{I_2}$ — " — $\quad I_2 = \mathbb{R} \setminus I_1$.



11/25 cont'd:

What if $L(p_i)=\infty$ for $p_i \in RHP$?

$\int_\Gamma \log S(s)\,ds = \lim_{\substack{R\to\infty \\ r\to 0}} I_1 + I_2 + I_3$



$I_2 = \int_{x^+} \log S(s)ds + \int_{x^-} \log S(s) ds + \int_\gamma \log S(s) ds.$

$= \int_{x^+} [\log|S(s)| + i\arg S(s)]\, ds + \int_{x^-} [\log|S(s)| + i\arg S(s)]\, ds$

cancel differ by 2π

$= 2\pi i \cdot Re(p_i)$

Gunter Stein → 1^{st} Bode Lecture

"Respect the unstable"

12/2:

Sample Robust performance criterion:

$\| |W_p S| + |W_r T| \|_\infty < 1$

weight big when we want S to be small

weight big when model is poor

T: d→u, S: r→e } places where these show up.

Limits:

i) $S + T = 1$.

$\Rightarrow \min(|W_p(i\omega)|, |W_r(i\omega)|) < 1 \quad \forall \omega.$

ii) $\int_0^\infty \ln|S(i\omega)|\,d\omega = \pi \sum_{p_i \in RHP \text{ for } L(s)} p_i$

All-pass transfer functions

$H(s)$ is all-pass if $|H(i\omega)| = 1 \ \forall \omega$.

Ex: $e^{-\tau s}$ $\quad \frac{s-z}{s+z}$

Theorem: For any $G(s)$ rational $\exists\, G_{mp}, G_{ap}$ rational s.t. $G = G_{mp}G_{ap}$, G_{mp} is min phase (no RHP poles/zeros) and G_{ap} is all pass).

Proof: suppose $G(s) = \frac{(s+z_1)\cdots(s+z_\ell)(s-z_{\ell+1})\cdots(s-z_m)}{(s+p_1)\cdots(s+p_k)(s-p_{k+1})\cdots(s-p_n)}$

$Re(z_i) > 0$; $Re(p_i) \le 0$

$G_{mp} = \frac{(s+z_1)\cdots(s+z_\ell)(s+z_{\ell+1})\cdots(s+z_m)}{(s+p_1)\cdots(s+p_k)(s+p_{k+1})\cdots(s+p_n)}$

$G_{ap} = \frac{(s-z_{\ell+1})}{(s+z_{\ell+1})} \cdots \frac{s-z_m}{s+z_m} \cdot \frac{s+p_{k+1}}{s-p_{k+1}} \cdots \frac{s+p_n}{s-p_n}.$

Theorem: Maximum modulus principle

For any non-constant function $F: \mathbb{C} \to \mathbb{C}$ that is analytic on Ω closed subset of $\mathbb{C}$ the maximum of $|F|$ cannot occur on the interior of Ω.

Corollary: Let $\Omega = \overline{RHP}$ (closed) and $H(s)$ transfer function w/ no RHP poles: $\|H\|_\infty = \max_{s \in \overline{RHP}} H(s) \ge H(s) \ \forall s \in RHP$

Simple Bounds:

$\|W_p S\|_\infty \ge |W_p(z)|$ for z, a RHP zero of $L(s) = P(s)C(s)$

↳ cannot get good performance around RHP zeros

$\|W_r T\|_\infty \ge |W_r(p)|$ for p, a RHP pole of $L(s)$

↳ can't get good robustness around RHP poles.

If $P(s)$ has pole in LHP at p_0, we can do a pole/zero cancellation

$P(s) = \frac{n_p(s)}{\delta_p(s)(s+p_0)}$ $\quad C(s) = \frac{n_c(s)}{\delta_c(s)} s+p_0$

"You can cancel all you want in the LHP".

· Suppose $L(s)$ has RHP zero at z and RHP pole @ p.

$S = \frac{1}{1+PC} = \frac{d_p d_c}{d_p d_c + n_p n_c}$ $\quad$ RHP zero @ p.

$S = S_{mp} \cdot \frac{s-p}{s+p} = S_{mp} \cdot S_{ap}$ $\quad S(z) = 1$; $S_{mp}(z) = S_{ap}(z)^{-1} = \frac{z+p}{z-p}$.

$\|W_p S\|_\infty = \|W_p S_{mp}\|_\infty \ge |W_p(z) S_{mp}(z)| = |W_p(z) \frac{z+p}{z-p}|$

RHP pole zero cancelations are not internally stabilizing (unless perfect)...? But they also cause terrible performance.

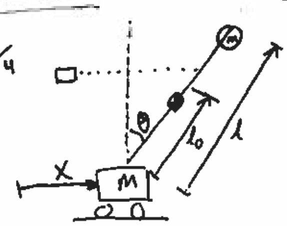
$\|W_r T\|_\infty \ge |W_r(p) \frac{p+z}{p-z}|$ $\quad$ p is RHP pole, z is RHP zero of $L(s)$.

☆ [Quals question: Suppose you have RHP pole/zero
↳ Bode integral/Maximum modulus principle
What are limits on performance?]

· Let $M_1 = \max_{\omega_1 \le \omega \le \omega_2} |S(i\omega)|$; $M_2 = \max_\omega |S(i\omega)| = \|S\|_\infty$

Theorem: $c_1 \log M_1 + c_2 \log M_2 \ge \log |S_{ap}(z)|^{-1}$: z RHP zero of $L(s)$.

12/4



$(M+m)\ddot{x} + ml(\ddot{\theta}\cos\theta - \dot{\theta}^2 \sin\theta) = u + r$

$m(\ddot{x}\cos\theta + l\ddot{\theta} - g\sin\theta) = 0$

$z = x_0 + l_0 \sin\theta$ $\quad$ output

$y = z + n$ $\quad$ measurement

r = actuation noise; n = measurement noise.

Poles: $0, \pm\sqrt{\frac{(M+m)g}{Ml}}$

Zeros: if $l_0 < l$: $\pm\sqrt{\frac{g}{l - l_0}}$

if $l_0 \ge l$: $\pm i\sqrt{\frac{g}{l_0 - l}}$

$\rightsquigarrow$ $p = z\sqrt{1+r}$ $\quad r = \frac{m}{M}$

$z = \sqrt{\frac{g}{l}}$

Interpretation: $l\uparrow$

$\|W_z T\|_\infty \ge |W_z(p) \frac{p+z}{p-z}|$

12/6: Review: "If you didn't have a HW, it won't be there"

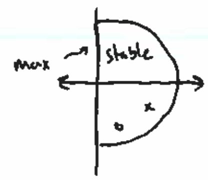
· Youla parameterization. Not as much algebra.

THE LIST:

- ?? · Maximum Principle
- X · Coprime factorization
- ✓ · Robust Performance
- ✓ · LQR
- ? · Linear observers
- ?? · Kalman decomposition
- ✓ · A invariant subspaces
- ?? · Youla parameterization
- X · Robust stability/performance with two uncertainties
- X · Induced norms
- ✓ · Observability tests
- ✓ · Stability tests
- X · Bode's integral formula
- X · MIMO small gain
- X · Chordal/ν-gap
- ✓ · Limits of performance
- ✓ · Controllability/observability Gramians
- X · Phase margin
- X · Bode plots
- ??? · Nyquist plots
- ✓ · Solution of linear input/output systems
- ? · Convolution
- ✓ · Block diagram [there's one in the exam]
- ? ↳ not a lot of block diagram algebra

Limits of Performance:

- Check first for stability
- Algebraic limits → always true
- Integral formulas [e.g. water bed effect]
  - ↳ "mostly won't ask about integral formulas"
- Maximum modulus principle → "know and be able to apply"
  - ↳ if stable & bounded, max will occur on $j\omega$ axis.



$S(z)=1 \quad S(p)=0 \quad S=\frac{1}{1+PC}$

$T(z)=0 \quad T(p)=1 \quad T=\frac{PC}{1+PC}$

$\|W_p S\|_\infty \geq W_p(s)S(s) \quad \forall s \in RHP.$

$\|H_{ap}\|_\infty = 1$

$\|H \cdot H_{ap}\|_\infty = \|H\|_\infty$

→ how to do all phase } manipulate problem ✩

→ min of H_∞

$S=\frac{1}{1+PC} = \frac{d_p d_c}{d_p d_c + n_p n_c} = S_{mp} \cdot \frac{s-p}{s+p} \rightarrow \frac{z-p}{z+p}$

$S = 1 = S_{mp} S_{ap}$

$S_{mp} = S_{ap}^{-1}$

$\|W_p S\|_\infty = \|W_p S_{mp}\|_\infty \geq W_p \left|\frac{z+p}{z-p}\right|$ → W_p evaluated at z.

RHP pole zero ⇒ using S_{ap} & factorization ✩

A-invariant Subspaces:

Definition: $V \subset \mathbb{R}^n \qquad Av \in V \quad \forall v \in V$

Fact: reachable space is the smallest A-invariant space containing B

Fact: unobservable space is the largest A-invariant space annihilated by C

↳ start with B. if it's invariant than AB must be in there ... Look at A^2B, etc. when this stops growing, it's the largest A invariant space containing B.

all of these are subspace → $E_{r o}$

well not really... → E_r → $E_{r\bar{o}}$ → only A-invariant smaller space

E_o → $E_{\bar{r}o}$

A invariant → $E_{\bar{r}\bar{o}}$

$E_{\bar{r}}$ is not a subspace, it is the complement to E_r

Reachable canonical form:

$A = \begin{bmatrix} -a_1 & \cdots & a_n \\ 1 & \ddots & 0 \\ & \ddots & \vdots \\ & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad$ Reachability matrix $= \begin{bmatrix} 1 & * & * \\ & \ddots & * \\ 0 & & 1 \end{bmatrix}$ → reachable

Not reachable ⇒ no transformation to reachable canonical form

↳ if reachability matrix is not invertible.

LQR:

$L = \int_0^T (x^T Q x + u^T R u)\,dt \qquad Q \geq 0, \; R > 0$

$-\dot{P} = PA + A^T P + PB^T R^{-1} B P, \quad P(T) = P_0$

$=0$ ↳ fell out from maximum principle

↳ Ricatti equation:

$u = -R^{-1} B^T P x$

→ stabilizing properties of LQR

→ Consequences of if Q & R are not ≻0

✩ Where do the equations come from, what do they mean, etc...?

Robust Performance: → where do these come from?

Robust performance ⟺ robust stability + performance → $\|W_p \tilde{S}\|_\infty \leq 1$ ($\tilde{S} = \frac{1}{1+\tilde{P}C}$)

↑ knowing this — ↑ can I manipulate this into some form? → ex: $\| |W_p S| + |W_T T| \|_\infty < 1$

Necessary: If this is satisfied, then I can find a controller

↳ show how you would find this.