Math 151a: Differential & Algebraic Topology

Lecture 1:

topology $\longrightarrow$ algebra

Top Invariants:

- fundamental group (fund gp) & higher homotopy (htpy) gps
- Homology (hlgy) and Cohomology (cohlgy)

Relationships: eg $\pi_1 \to H_1$, abelianization

Going off of $H^n = (H_n)^\vee$.

Thm (Poincaré Duality): $M = M^d$, a closed, oriented d-manifold (mfld)

$$H^i(M) \cong H_{d-i}(M).$$

Homology: $H_n : \mathrm{Top} \longrightarrow \mathrm{Ab}$ $\forall$ $n \geq 0$

Three constructions:

$$\begin{array}{lll} \Delta\text{-complexes} \rightsquigarrow \text{simplicial hlgy} & & \text{simplest, combinatorial} \\ \quad \cap \qquad\qquad\quad \cong & & \\ \text{CW-complexes} \rightsquigarrow \text{cellular hlgy} & & \text{most efficient} \\ \quad \cap \qquad\qquad\quad \cong & & \\ \text{top spaces} \rightsquigarrow \text{singular hlgy} & & \text{most general, fully functorial} \end{array}$$

(any cts function $X \xrightarrow{f} Y$ gives $H_n(X) \xrightarrow{H_n(f)} H_n(Y)$)

Aside: Hlgy is *characterized* by Eilenberg Steenrod axioms (Sec 2.3). Relax "dimension axiom": get *extraordinary* hlgy thms (e.g. K-theory, cobordism)

Topological Applications:

(1) Borsak-Ulam thm: $\forall$ cts $S^n \xrightarrow{f} \mathbb{R}^n$ $\exists\, x \in S^n$ s.t. $f(x) = f(-x)$

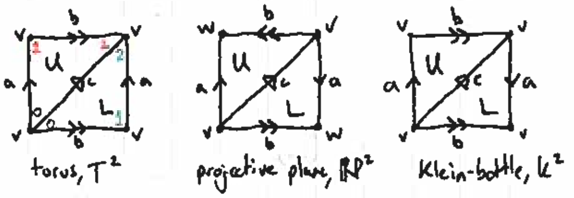
(2) Lefschetz fixed point theorem: $X \in \mathrm{Top}^{fin}$ (sufficiently finite), any $X \xrightarrow{f} X$, its *Lefschetz number* is: $\tau(f) := \sum_{n \in \mathbb{Z}} (-1)^n \cdot tr(\underbrace{H_n(f)}_{H_n(x) \to H_n(x)})$ $(\in \mathbb{Z})$

Thm: If $\tau(f) \neq 0$, then f has a fixed point.

(3) Hairy ball theorem: S^n admits a nonvanishing (continuous) vector field iff n is *odd*.

Simplicial Homology

Ex of Δ-complexes:



☆ Each simplex has a total ordering of its vertices. ("n-dimensional triangle") and those of faces $\vee$ of a simplex are inherited

Orientation of a simplex

Disallowed:


(Topological n-simplex) $\Delta^n_{(top)} := \{ t = (t_0, \dots, t_n) \in \mathbb{R}^{n+1} : t_i \geq 0, \sum t_i = 1 \}$.

Def: A *chain complex* (of ab. groups) is a sequence of abelian groups & hom's:

$$C_\bullet = (\dots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \xrightarrow{\partial_{n-2}} \dots)$$

such that "$\partial^2 = 0$" i.e. $\forall$ $n \in \mathbb{Z}$, $\partial_n \circ \partial_{n+1} = 0$

Equivalently, $\mathrm{im}(\partial_{n+1}) \subseteq \ker(\partial n) \subseteq C_n$. Say $C_\bullet$ is *exact* in dim n if $\mathrm{im}(\partial_{n+1}) = \ker(\partial_n)$

So, n^{th} homology measures *failure of exactness*:

$$H_n(C_\bullet) = \frac{\ker(\partial n)}{\mathrm{im}(\partial_{n+1})} \begin{array}{l} \} Z_n(C_\bullet) \text{ n-cycles} \\ \} B_n(C_\bullet) \text{ n-boundaries} \end{array}$$

$\mathrm{Ch} = \mathrm{Ch}_{\mathbb{Z}}$, abelian chain complexes

$$\begin{tikzcd} \dots C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"'] \arrow[dr, phantom, "\circlearrowleft"] & C_{n-1} \dots \arrow[d, "f_{n-1}"] \\ \dots D_n \arrow[r] & D_{n-1} \dots \end{tikzcd}$$

Sequence commutes: $f_{n+1} \circ \partial_n^C = \partial_n^D \circ f_n$

Def: For a Δ-cx X, its chain complex of simplicial chains

$$C^\Delta_\bullet(X) := (C_n := \bigoplus_{\substack{n\text{-simplices} \\ \text{of } X}} \mathbb{Z},\ C_n \xrightarrow{\partial_n} C_{n-1})$$

$\boxed{\begin{array}{c} (n \geq 0) \\ C^\Delta_{<0}(x) = 0 \end{array}}$

on generators: $\underset{\text{an n-simplex of } X}{\sigma} \longmapsto \sum_{i=0}^n (-1)^i \underline{\hat{\delta}_i(\sigma)}$

ith face of σ obtained by removi ith vertex.

$$\dots \to C^\Delta_1(x) \to C^\Delta_0(x) \to 0 \to 0 \to \dots$$

Lemma: $\partial^2 = 0$.

Def: Simplicial homology is $H^\Delta_n(X) := H_n(C^\Delta_\bullet(x))$.

Ex: $C^\Delta_\bullet(T^2) = \left(C^\Delta_2(T^2) \to C^\Delta_1(T^2) \to \underline{C^\Delta_0(T^2)} \right)$

all other terms zero.

$= \left(\mathbb{Z}\{U, L\} \to \mathbb{Z}\{a,b,c\} \to \mathbb{Z}\{v\} \right)$

$U \mapsto \delta_0(U) - \delta_1(U) + \delta_2(U)$
$= b - c + a$
$L \mapsto a - c + b$

$a \longmapsto \delta^1_0(a) - \delta^1_1(a)$
$=$ target of a $-$ source of a
$= v - v$
$= 0$

$b \longmapsto v - v = 0$
$c \longmapsto v - v = 0$
$\Rightarrow a, b, c \in \ker(\partial_1)$
$=: Z_1(C^\Delta_1(X))$
1-cycles

$H^\Delta_2(T^2) = \frac{\ker(\partial_2)}{\operatorname{im}(\partial_3)} = \ker(\partial_2)$ ①
$= \mathbb{Z}\{U - L\} \cong \mathbb{Z}$

$H^\Delta_1(T^2) = \frac{\ker(\partial_1)}{\operatorname{im}(\partial_2)} = \frac{\mathbb{Z}\{a,b,c\}}{\mathbb{Z}\{b-c+a\}}$ ②
$\cong \mathbb{Z}\{a, b\} \cong \mathbb{Z} \oplus \mathbb{Z}$

$H^\Delta_0(T^2) = \frac{\ker(\partial_0)}{\operatorname{im}(\partial_1)} = \frac{\mathbb{Z}\{v\}}{0} \cong \mathbb{Z}.$ ①

#: Dimension → Reflection of Poincaré Duality.

Lecture 2:

Idea: A Δ complex is:
→ (abstract) A recipe for building a topological space out of simplices $(\Delta^n =: \Delta^n_{Top} \subset \mathbb{R}^{n+1} \dots)$
→ A top space equipped with a decomposition into simplices glued with standard maps b/w them.

Ex: A 2-dim'l (abstract) Δ-cx $X_\bullet$ is:
↳ sets X_0, X_1, X_2 of vertices, edges, faces.
↳ identification of the $(n-1)$ simplices that are the faces of n-simplices, i.e. functions

$$\begin{tikzcd} X_2 \arrow[r, shift left=2, "\delta^2_0"] \arrow[r, "\delta^2_1" description] \arrow[r, shift right=2, "\delta^2_2"'] & X_1 \arrow[r, shift left=1, "\delta^1_0"] \arrow[r, shift right=1, "\delta^1_1"'] & X_0 \end{tikzcd}$$

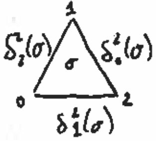
th face", i.e.
e span of
ll the
rtices except
the 0th

1st face

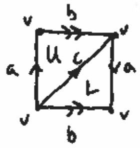
2nd face

"0th face" function a.k.a "ending vertex"

starting vertex



Ex: Klein bottle



$X_2 = \{U, L\}$
$X_1 = \{a, b, c\}$
$X_0 = \{v\}$

such that: $\begin{cases} \delta^1_0 \delta^2_0 = \delta^1_0 \delta^2_1 \\ \delta^1_1 \delta^2_0 = \delta^1_0 \delta^2_2 \\ \delta^1_1 \delta^2_1 = \delta^1_1 \delta^2_2 \end{cases}$

Chain Complex:

$$C^\Delta_\bullet(K^2) = \left(\mathbb{Z}\{U, L\} \xrightarrow{\partial_2} \mathbb{Z}\{a,b,c\} \xrightarrow{\partial_1} \mathbb{Z}\{v\} \right)$$

$$\partial_n = \sum_{i=0}^{n} (-1)^i \delta^n_i$$

For $C_\bullet = (\dots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \dots) \in Ch$ by definition

$\underset{B_n(C_\cdot)}{\underset{\shortparallel}{\operatorname{im}(\partial_{n+1})}} \subseteq \underset{Z_n(C_\cdot)}{\underset{\shortparallel}{\ker(\partial_n)}}$

$$H_n(C_\cdot) = \frac{Z_n(C_\cdot)}{B_n(C_\cdot)} = \frac{\ker(\partial_n)}{\operatorname{im}(\partial_{n+1})}$$

Simplicial homology: $H^\Delta(X) = H(C^\Delta(X))$

Homology groups for K^2:

$U \longmapsto b - c + a$
$L \longmapsto a - b + c$

$a \longmapsto 0$
$b \longmapsto 0$
$c \longmapsto 0$

$H^\Delta_0(K^2) = \frac{\ker(\partial_0)}{\operatorname{im}(\partial_1)} = \frac{\mathbb{Z}\{v\}}{0} \cong \mathbb{Z}$

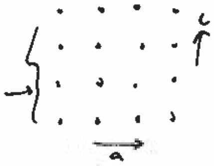
$H^\Delta_2(K^2) = \frac{\ker(\partial_2)}{\operatorname{im}(\partial_3)} = \ker(\partial_2) = \{x \cdot U + y \cdot L : \partial_2(x \cdot U + y \cdot L) = 0$ $x, y \in \mathbb{Z}$
hom $\Rightarrow x \cdot \partial_2(U) + y \cdot \partial_2(L) = 0$
$\Leftrightarrow (x+y)a + (x-y)b + (-x+y)c = 0$
$\Leftrightarrow x = y = 0$
$= 0.$

Even over $\mathbb{Z}$
$\begin{matrix} x = 0 \\ y = 0 \end{matrix} \Leftrightarrow \begin{matrix} x = 0 \\ x + y = 0 \end{matrix}$

$H^\Delta_1(K^2) = \frac{\ker(\partial_1)}{\operatorname{im}(\partial_2)} = \frac{\mathbb{Z}\{a,b,c\}}{\mathbb{Z}\{b-c+a, a-b+c\}}$
$= \frac{\mathbb{Z}\{a,b,c\}}{\mathbb{Z}\{b-c+a, 2a\}} = \frac{\mathbb{Z}\{a,c\}}{\mathbb{Z}\{2a\}} \cong \mathbb{Z}/2 \oplus \mathbb{Z}$

Visualization

$\mathbb{Z}\{a, c\} \to$



Can move freely in c, 2a in a direction, hence $\mathbb{Z}/2 \oplus \mathbb{Z}$.

$\Delta :=$ the category $\begin{cases} ob = \{[0], [1], \dots \rightsquigarrow [n] = \{0 < 1 < \dots < n \\ mor = \text{order-preserving functions.} \end{cases}$

the subcategory (same objects & morphisms)

$$\begin{tikzcd} \Delta_{\le 2} : & {[2]} & {[1]} \arrow[l, shift left=2, "d^2_2"'] \arrow[l, "d^2_1" description] \arrow[l, shift right=2, "d^2_0"] & {[0]} \arrow[l, shift right=1, "d^1_0"'] \arrow[l, shift left=1, "d^1_1"] \end{tikzcd}$$

$\{0 < 1 < 2\}$ $\{0 < 1\}$ $\{0\}$

$1 \leftarrow\!\shortmid 0$
$= d^2_2 d^1_0 = d^2_0 d^1_1$

$\Delta_{inj} \subset \Delta$: same objects, but only injective order preserving fns.

reverse all morphisms

A 2-dimensional Δ-complex is a functor $(\Delta_{inj, \le 2})^{op} \xrightarrow{X_\bullet} Set$

A Δ-complex is a functor $(\Delta_{inj})^{op} \to Set$

Set
$[n] \mapsto \emptyset$
$\forall n \ge 3$

$\Delta C_X = Fun((\Delta_{inj})^{op}, Set).$

L3:

Recall: $\Delta_{inj} := \begin{cases} \text{ob}: [0]=\{0\}, [1]=\{0,1\}, \dots \\ \text{mor}: \text{order preserving functions (inj.)} \end{cases}$

A Δ-complex is a functor $\Delta_{inj}^{op} \xrightarrow{X_\bullet} Set$

(X_n := the set of n-simplices of $X_\bullet$)

$\Delta Cx := Fun(\Delta_{inj}^{op}, Set)$

$$\Delta_{inj} = \left(\begin{tikzcd} {[0]} \arrow[r, shift left, "d_0^1"] \arrow[r, shift right, "d_1^1"'] & {[1]} \arrow[r, shift left=2, "d_0^2"] \arrow[r, "d_1^2" description] \arrow[r, shift right=2, "d_2^2"'] & {[2]} \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \cdots \end{tikzcd} \right)$$

d_i^n = unique injective fxn that misses $i \in [n]$

$$\Delta_{inj}^{op} = \left(\begin{tikzcd} {[0]^\circ} & {[1]^\circ} \arrow[l, shift right, "\delta_0^1"'] \arrow[l, shift left, "\delta_1^1"] & {[2]^\circ} \arrow[l, shift right=2, "\delta_0^2"'] \arrow[l, "\delta_1^2" description] \arrow[l, shift left=2, "\delta_2^2"] & \cdots \arrow[l, shift right=3] \arrow[l, shift right=1] \arrow[l, shift left=1] \arrow[l, shift left=3] \end{tikzcd} \right)$$

generators: obj's $[n]^\circ$, mor's δ_i^n

Relations: $\delta_j^n \delta_i^{n+1} = \delta_i^n \delta_{j+1}^{n+1}$ for $0 \leq i \leq j \leq n$

Given a Δ-complex $X_\bullet$, its ch cx of simpl chains (w/ coeffs in $\mathbb{Z}$)

$$C_\bullet^\Delta(X_\bullet) := \left(\cdots \xrightarrow{\partial_{n+1}} C_n^\Delta(X_\bullet) \xrightarrow{\partial_n} C_{n-1}^\Delta(X_\bullet) \to \cdots \xrightarrow{\partial_1} \underline{C_0^\Delta(X_\bullet)} \to 0 \right)$$

$$\begin{tikzcd} C_n^\Delta(X_\bullet) \arrow[r, "\partial_n"] \arrow[d, equal] & C_{n-1}^\Delta(X_\bullet) \arrow[d, equal] \\ \mathbb{Z}\{X_n\} \arrow[r] & \mathbb{Z}\{X_{n-1}\} \end{tikzcd}$$

$\partial_n := \sum_{i=0}^{n} (-1)^i \cdot \delta_i^n$

Claim: $\partial^2 = 0$ (i.e. $\partial_n \circ \partial_{n+1} = 0$)

Proof: Suffices to check on a generator, i.e. $\sigma \in X_{n+1}$ considered in $C_{n+1}^\Delta(X_\bullet) := \mathbb{Z}\{X_{n+1}\}$

$$\partial_n(\partial_{n+1}(\sigma)) := \partial_n\left(\sum_{i=0}^{n+1} (-1)^i \delta_i^{n+1}(\sigma)\right) := \sum_{i=0}^{n+1} (-1)^i \partial_n(\delta_i^{n+1}(\sigma))$$

$$:= \sum_{i=0}^{n+1} (-1)^i \Big(\sum_{j=0}^{n} (-1)^j \underbrace{\delta_j^n \delta_i^{n+1}(\sigma)}_{\substack{\text{(n-1) simplices in } X_\bullet \\ \text{the various boundaries of } \sigma}} \Big) = \begin{cases} i \leq j \to \text{apply relation} \\ i > j \to \text{don't} \end{cases}$$

$= 0$

→ these each show up twice with opp sign ■

Idea: A Δ-cx is a recipe for building a topological space

Def: The geometric realization of $X_\bullet \in \Delta Cx$ is

$$|X_\bullet| := \Big(\coprod_{n \geq 0} \overset{\text{i.e. } \coprod_{X_n} \Delta^n}{\overbrace{\underbrace{\Delta^n}_{\text{the top. n-simplices}} \times X_n}} \Big) \Big/ \forall n, \forall 0 \leq i \leq n+1, \forall \sigma \in X_{n+1}$$

$\Delta^n := \{\tau \in \mathbb{R}^{n+1} : \tau_i \geq 0, \Sigma \tau_i = 1\}$

$$\begin{tikzcd} \Delta^n \times \{\delta_i^{n+1}(\sigma)\} \arrow[d, phantom, "\cong" rotate=90] & \Delta^{n+1} \times \{\sigma\} \arrow[d, phantom, "\cong" rotate=90] \\ \Delta^n \arrow[d, phantom, "\ni" rotate=90] & \Delta^{n+1} \arrow[d, phantom, "\ni" rotate=90] \\ x \arrow[r, dash] & d_i^{n+1}(x) \end{tikzcd}$$

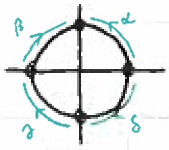
$$= X_\bullet \underset{\Delta_{inj}}{\otimes} \Delta^\bullet$$

$\Delta^n \xrightarrow{d_i^{n+1}} \Delta^{n+1}$

the fxn: $[n] \xrightarrow{d_i^{n+1}} [n+1]$ on vertices, extended linearly

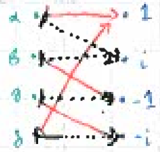
A $\underline{\Delta\text{-cx structure}}$ on a top spc. T is: a Δ-cx $X_\bullet \in \Delta Cx$ and a homeo $|X_\bullet| \cong T$

Ex: Problem 3, dim 1:



$$\begin{tikzcd} S^1 \subset \mathbb{R}^2 \arrow[d, "\substack{\text{quotient by} \\ \mathbb{Z}/2\text{, action} \\ p \mapsto -p}"] \\ \mathbb{RP}^1 \end{tikzcd}$$

$$\begin{tikzcd} X_1 \arrow[r, shift left, "\delta_0^1"] \arrow[r, shift right, "\delta_1^1"'] & X_0 \end{tikzcd}$$



Simplicial homology: "easy" to compute

↳ Q: Given two Δ-cx structures on a top space, do the H_n^Δ's agree?

↳ Q: Given a cts fxn $X \to Y$, how do we get a hom $H_n^\Delta(X) \to H_n^\Delta(Y)$ if it doesn't respect Δ-cx structures?

Construction: Given a top space X, its singular complex is the Δ-cx

$$Sing_n(X) := hom_{Top}(\Delta^n, X) := \{\Delta^n \to X \text{ cts}\}$$

Structure maps come from:

$$\Delta_{inj} \xrightarrow{\Delta^\bullet} Top \rightsquigarrow Sing(X)_\bullet : \Delta_{inj}^{op} \to Top^{op} \xrightarrow{hom_{Top}(-, X)} Set$$

$[n] \mapsto \Delta^n$

"$\boxed{\text{the convex hull of elts of } [n]}$"

Concretely,

$$\delta_i^n\big((\Delta^n \xrightarrow{\sigma} X)\big) := (\Delta^{n-1} \xrightarrow{d_i^n} \Delta^n \xrightarrow{\sigma} X)$$

$\overset{\wedge}{Sing(X)_n}$

L4: Singular Homology

$$\begin{tikzcd}
\Delta Cx \arrow[r, "1\text{-}1"] \arrow[d] \arrow[dd, bend right, "H_n^\Delta(-)"'] & Top \arrow[l, dotted, bend left, "Sing_\bullet(-)"] \arrow[dl, bend left, "C_\bullet^{sing}(-)"] \arrow[ddl, bend left, "H_n^{sing}(-)"] \\
Ch \arrow[d] & \\
Ab &
\end{tikzcd}$$

geometric realization

$Sing(X)_n := \hom_{Top}(\Delta^n, X)$
$:\{$cts fncs from $\Delta^n \to X\}$

Def: For $X \in Top$, its ch cx of <u>singular chains</u> is:

$$C_\bullet^{sing}(X) := \Big(\overbrace{\mathbb{Z}\{\underbrace{\hom_{Top}(\Delta^n, X)}\}}^{n\text{th } (n\geq 0)}, C_n^{sing}(X) \xrightarrow{\partial_n} C_{n-1}^{sing}(X)\Big)$$

the hom-set in the category Top

Set of cts fncs ("singular n-simps in X")

free ab grp on ↑

$C_n^{sing}(X) = \mathbb{Z}\{\hom_{Top}(\Delta^n, X)\}$, $\quad C_{n-1}^{sing}(X) = \mathbb{Z}\{\hom_{Top}(\Delta^{n-1}, X)\}$

generator $(\Delta^n \xrightarrow{\sigma} X) \mapsto \sum_{i=0}^{n} (-1)^i \underbrace{\delta_i^n(\sigma)}$

$(\Delta^{n-1} \xrightarrow{d_i^n} \Delta^n \xrightarrow{\sigma} X)$

Q: Why is this a chain cx, and why do cts fncs go to chain maps?

A: Really, it's a composite functor

$$\Delta_{inj} \xrightarrow{\Delta^\bullet} Top \qquad \rightsquigarrow \qquad C_\bullet^{sing}(X) : \Delta_{inj}^{op} \xrightarrow{\Delta^\bullet} Top^{op} \xrightarrow{\hom_{Top}(-,X)} Set$$

$$[n] \longmapsto \Delta^n$$

e.g.

$$\begin{tikzcd}
{[n-1]} \arrow[d, "d_i^n"'] & \longrightarrow & \Delta^{n-1} \arrow[d, "d_i^n"] \\
{[n]} & & \Delta^n
\end{tikzcd}
\qquad
\begin{tikzcd}
{[i]} \arrow[d, "\varphi"'] & \hom_{Top}(\Delta^i, X) \\
{[j]} \arrow[r, mapsto] & \hom_{Top}(\Delta^j, X) \arrow[u, "\text{precomp w/ } \Delta^i \xrightarrow{\varphi} \Delta^j"']
\end{tikzcd}$$

skip the elt $i \in [n]$

Prop (2.6): If $X \cong \coprod_{\alpha\in A} X_\alpha$, then $H_n^{sing}(X) \cong \bigoplus_{\alpha\in A} H_n^{sing}(X_\alpha)$

Proof: In fact, $C_\bullet^{sing}(X) \cong \bigoplus_{\alpha\in A} C_\bullet^{sing}(X_\alpha)$.

i.e.

$$\begin{tikzcd}
C_n^{sing}(X) \arrow[d, "\partial_n^X"'] \arrow[r, phantom, "\cong"] & \bigoplus_{\alpha\in A} C_n^{sing}(X_\alpha) \arrow[d, dotted, "\bigoplus_{\alpha\in A} \partial_n^{X_\alpha}"] \\
C_{n-1}^{sing}(X) \arrow[r, phantom, "\cong"] & \bigoplus_{\alpha\in A} C_{n-1}^{sing}(X_\alpha)
\end{tikzcd}$$

diagonal matrix (no cross terms)

$$\sum_{\beta\in B} \overset{\in\mathbb{Z}}{n_\beta} \cdot \overset{\in \hom_{Top}(\Delta^n, X)}{\sigma_\beta} \longleftrightarrow \Big(\sum_{\beta\in B_\alpha} n_\beta \cdot \sigma_\beta\Big)_{\alpha\in A}$$

B a finite indexing set / For each $\alpha \in A$, write $B_\alpha \subseteq B$ for the subset $\{\beta \in B : \Delta^n \xrightarrow{\sigma_\beta} X$ factors through $X_\alpha\}$

Note: $B = \coprod_{\alpha\in A} B_\alpha$ (because $\Delta^n \in Top$ is connected).

∴ b/c if $\sigma \in C_n^{sing}(X)$ lands in X_α, then so do all $\delta_i^n \circ \sigma \in C_{n-1}^{sing}(X)$

Lemma: $Ch \xrightarrow{H_n} Ab$ preserves direct sums.

Prop (2.7): For $X \in Top$ nonempty and path-connected, $H_0(X) \cong \mathbb{Z}$.

Pf:

$$\begin{tikzcd}
C_1(X) \arrow[r, "\partial"] \arrow[d, equal] & C_0(X) \arrow[r, "\partial_0 = 0"] \arrow[d, equal] & 0 \\
\mathbb{Z}\{\text{paths in } X\} & \mathbb{Z}\{\text{points in } X\} \arrow[d, "\varepsilon"] & \\
& \mathbb{Z} &
\end{tikzcd}$$

"sum of coefficients"

$\ni \sum_{\beta\in B} n_\beta \sigma_\beta \mapsto \sum_{\beta\in B} n_\beta \in \mathbb{Z}$

$\sigma_\beta \in \hom_{Top}(\Delta^0, X)$ underlying set of X

Claim: the sequence $C_1(X) \xrightarrow{\partial} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z}$ is <u>exact at $C_0(X)$</u>, i.e. $\operatorname{im}(\partial_1) = \ker(\varepsilon)$.

Then, observe ε is surjective, so (b/c X is nonempty)

$$H_0(X) := \frac{\ker(\partial_0)}{\operatorname{im}(\partial_0)} =: \frac{C_0(X)}{\operatorname{im}(\partial_1)} \underset{!}{=} \frac{C_0(X)}{\ker(\varepsilon)} \xrightarrow{\cong} \mathbb{Z}$$

proving claim:

$\subseteq$: For any generator (cts fncs $\overset{[0,1]}{\Delta^1} \xrightarrow{\gamma} X$) $\gamma \in C_1(X)$,

$\varepsilon(\partial_1(\gamma)) := \varepsilon(\gamma(1) - \gamma(0)) := 1 - 1 = 0$ ✓

$\supseteq$: If $\sum_i n_i \sigma_i \in C_0(X)$ has $\varepsilon(\sum_i n_i \sigma_i) = 0$, then we can decompose it as a sum of terms of the form $(\sigma_i - \sigma_{i'})$

b/c X is path connected, ∃ path γ from $\sigma_{i'}$ to σ_i, and so $\partial_1(\gamma) = \sigma_i - \sigma_{i'}$. So $(\sigma_i - \sigma_{i'}) \in \operatorname{im}(\partial_1)$, so $\sum_i n_i \sigma_i \in \operatorname{im}(\partial_1)$ ∎

Corr: (2.6+2.7): ∀ $X \in Top$

$$H_0(X) \cong \bigoplus_{\substack{\text{path} \\ \text{components of } X}} \mathbb{Z}$$

L5: Singular Homology cont'd:

Prop 2.8: $H_n(pt) \cong \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n > 0 \end{cases}$

∄ Δ-cx str on pt.

Pf: $\hom_{Top}(\Delta^n, pt) = \{\text{const}\}$, i.e. $sing(pt)$ has exactly one n-simplex ∀ $n \geq 0$,

$H_\bullet^\Delta(pt)$ Δ-cx: $X_0 \leftleftarrows X_1 \leftleftarrows X_2$, with $X_0 = \{pt\}$, $X_1 = \emptyset$, $X_2 = \emptyset$

$$\rightsquigarrow C_\bullet^\Delta(pt) = (\cdots \to \underset{0}{\mathbb{Z}\{X_2\}} \to \underset{0}{\mathbb{Z}\{X_1\}} \to \underset{\mathbb{Z}}{\underline{\mathbb{Z}\{X_0\}}}$$

i.e. $C_\bullet^\Delta(pt) = (\mathbb{Z})$

So $H_n^\Delta(pt) = \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n > 0 \end{cases}$

Call it $\sigma_n : \Delta^n \xrightarrow{!} pt$

So, $C_\bullet^{sing}(pt) := (\cdots \to \mathbb{Z}\{C_2^{sing}(pt)\} \to \mathbb{Z}\{C_1^{sing}(pt)\} \cdots$

$= (\cdots \to \mathbb{Z}\{\sigma_2\} \to \mathbb{Z}\{\sigma_1\} \to \mathbb{Z}\{\sigma_0\}$

differentials for $n \geq 0$:

$$\partial_n(\sigma_n) := \sum_{i=0}^{n} (-1)^i \cdot \delta_i^n(\sigma_n) = \sum_{i=0}^{n} (-1)^i \cdot \sigma_{n-1} = \begin{cases} 1 \cdot \sigma_{n-1}, & n \\ 0, & n \text{ odd} \end{cases}$$

$$= (\cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z})$$

So, $H_0^{sing}(pt) := \frac{\ker(\partial_0)}{\operatorname{im}(\partial_1)} = \frac{\mathbb{Z}}{0} = \mathbb{Z}$

$H_1^{sing}(pt) := \frac{\ker(\partial_1)}{\operatorname{im}(\partial_2)} = \frac{\mathbb{Z}}{\mathbb{Z}} = 0$

$H_2^{sing}(pt) := \frac{\ker(\partial_2)}{\operatorname{im}(\partial_3)} = \frac{0}{0} = 0$

$\vdots$

$H_{odd \geq 1} = 0$

$H_{even \geq 2} = 0$ ∎

L5 cont'd) Prop: $H_n^{sing}(\emptyset) = 0 \ \forall\ n$

Def: Recall: For any $X \in Top$, we have an <u>augmented</u> ch-cx

$$\tilde{C}_\bullet^{sing}(X) := \left(\cdots \to C_1^{sing}(X) \to C_0^{sing}(X) \xrightarrow{\overset{\text{sum of coeff's}}{\varepsilon}} \mathbb{Z} \right)$$

<u>Reduced homology</u> is $\tilde{H}_n^{sing}(X) := H_n(\tilde{C}_\bullet^{sing}(X))$.

So, $H_n \cong \tilde{H}_n$ for $n \geq 1$ $H_0 \cong \tilde{H}_0 \oplus \mathbb{Z}$

(A different "normalization" of homology:

$H_*(\emptyset) = 0$ whereas $\tilde{H}_*(pt) = 0$

<u>Homotopy Invariance:</u>

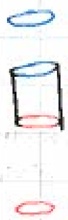
Observe the functoriality: $Top \xrightarrow{H_n^{sing}} Ab$, in particular $\forall\ X \xrightarrow{f} Y$, in Top, we get $H_n^{sing}(X) \xrightarrow{H_n^{sing}(f)} H_n^{sing}(Y)$ in Ab.

<u>Thm:</u> If $f, g : X \to Y$ are homotopic, then

$$\underbrace{H_n(f) = H_n(g)}_{\text{as hom's } H_n(X) \to H_n(Y)}$$

Def: homotopy: $\exists\ H : I \times X \to Y$ s.t.

$H(0,-) = f$, $H(1,-) = g$



$$\begin{tikzcd} X \times \{0\} \cong X \arrow[d, hook] \arrow[dr, "f"] & \\ X \times I \arrow[r, dashed, "H"] & Y \\ X \times \{1\} \cong X \arrow[u, hook'] \arrow[ur, "g"'] & \end{tikzcd}$$

(idea: a cts family of cts funct's $X \to Y$)
starting at f & ending at g

$$\begin{tikzcd} X \arrow[r, bend left=50, "f"{name=F}] \arrow[r, bend right=50, "g"'{name=G}] & Y \arrow[Rightarrow, from=F, to=G, "H"'] \end{tikzcd}$$

Def:

Given $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$ in Ch, a (chain) homotopy b/w them is:

$$P_\bullet := \left\{ \left(C_n \xrightarrow{P_n} D_{n+1} \right) : \overset{\forall n,}{\partial_{n+1}^D \circ P_n + P_{n-1} \circ \partial_n^C = g_n - f_n} \right\}$$

$$\begin{tikzcd} C_{n+1} & C_n \arrow[r, "\partial_n^C"] \arrow[dl, "P_n"'] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] & C_{n-1} \arrow[dl, "P_{n-1}"] \\ D_{n+1} \arrow[r, "\partial_{n+1}^D"'] & D_n & \end{tikzcd}$$

Prop: If $f_\bullet$ and $g_\bullet$ are chain-homotopic, then

$$\underbrace{H_n(f_\bullet) = H_n(g_\bullet)}_{\text{as hom's } H_n(C_\bullet) \to H_n(D_\bullet)} \quad \forall\ n \in \mathbb{Z}$$

Pf: $H_n(C_\bullet) := \dfrac{Z_n(C_\bullet)}{B_n(C_\bullet)} \leftarrow Z_n(C_\bullet) \ni x.$

$$\begin{array}{l} x \\ \downarrow \\ [x] \end{array} \left\{ \begin{array}{l} \forall\ x \text{ wts} \\ H_n(f_\bullet)([x]) = H_n(g_\bullet)([x]). \\ \qquad \overset{\shortparallel}{[f_n(x)]} \qquad\qquad \overset{\shortparallel}{[g_n(x)]} \end{array} \right.$$

Note:

$$\begin{tikzcd} Z_n(C_\bullet) \arrow[r, hook] \arrow[d, dotted] & C_n \arrow[d, "f_n"] \\ Z_n(D_\bullet) \arrow[r, hook] & D_n \end{tikzcd}$$

Equivalently, WTS $[g_n(x)] - [f_n(x)] = 0$.

$$\overset{!}{\shortparallel}$$

$$[g_n(x) - f_n(x)]$$

True iff $g_n(x) - f_n(x) \in B_n(D_\bullet) := im(\partial_{n+1}^D)$. And indeed,

$g_n(x) - f_n(x) =: (g_n - f_n)(x) = (\partial_{n+1}^D \circ P_n + P_{n-1} \circ \partial_n^C)(x)$

$:= \partial_{n+1}^D(P_n(x)) + P_{n-1}(\underbrace{\partial_n^C(x)}_{=0 \text{ since } x \in Z_n(C_\bullet)})$

$= \partial_{n+1}^D(\underbrace{P_n(x)}_{\in D_{n+1}})$ ▨

Pf of thm: Given htpy $f \Rightarrow g$, by Prop, it suffices to define a chain homotopy $C_\bullet(f) \Rightarrow C_\bullet(g)$. Roughly, to each

$$\underset{\text{(generator)}}{(\Delta^n \xrightarrow{\sigma} X)} \in C_n(X) := \mathbb{Z}\{hom_{Top}(\Delta^n, X)\}$$

we want to set

$$P_n(\sigma) :\approx (\Delta^n \times I \xrightarrow{\sigma \times id} X \times I \xrightarrow{H} Y) \text{“}\in\text{”} C_{n+1}(Y).$$

But $\Delta^n \times I$ isn't an $(n+1)$-simplex!

Rather, it is a union of many $(n+1)$-simplices

We'll set $P_n(\sigma) := \left\{ \begin{array}{l} \text{signed sum of} \\ \text{the } (n+1)\text{-simplices} \end{array} \right\}$

$\in C_{n+1}(Y) := \mathbb{Z}\{hom_{Top}(\Delta^{n+1}, Y)\}$.

L6:

Homotopy Invariance Continued:

Thm:
$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "H\Downarrow"] & Y \end{tikzcd}$$
in Top $\rightsquigarrow$
$$\begin{tikzcd} H_*(X) \arrow[r, bend left, "H_*(f)"] \arrow[r, bend right, "H_*(g)"'] & H_*(Y) \end{tikzcd}$$
(in gr Ab).

Cont'd $\searrow$
$$\begin{tikzcd} C_*(X) \arrow[r, bend left, "C_*(f)"] \arrow[r, bend right, "C_*(g)"'] \arrow[r, phantom, "\Downarrow P"] & C_*(Y) \end{tikzcd}$$
in Ch $\nearrow$ last time

Chain htpy: $\{C_n(X) \xrightarrow{P_n} C_{n+1}(Y)\}$: $\forall n \in \mathbb{Z}$
$\partial P + P \partial = C_n(g) - C_n(f)$

Pf idea: Want to assign

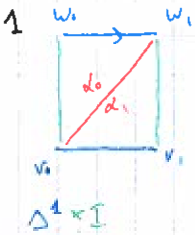
$$C_n(X) \ni (\Delta^n \xrightarrow{\sigma} X) \longmapsto p_n(\sigma) :\approx (\underbrace{\Delta^n \times I}_{\text{not an } (n+1) \text{ simplex!}} \xrightarrow{\sigma \times id} X \times I \xrightarrow{H} Y) \text{``}\in\text{''} C_{n+1}(Y)$$

Rather: Subdivide $\Delta^n \times I$ into $(n+1)$-simplices, then take the signed sum of these

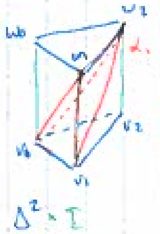
write $v_i := (i^{th} \text{ vertex}, 0)$ $\in \Delta^n \times I$
$w_i := (i^{th} \text{ vertex}, 1)$ $\in \Delta^n \times I$

$\rightsquigarrow \forall\ 0 \le i \le n$, get an $(n+1)$-simplex in $\Delta^n \times I$ on the span
$$\alpha_i := [v_0, \dots, v_i, w_i, \dots, w_n]$$

n=1



n=2



Set $P_n(\sigma) := \sum_{i=0}^{n} (-1)^i \cdot (\Delta^{n+1} \xrightarrow{\alpha_i} \Delta^n \times I \xrightarrow{\sigma \times id} X \times I \xrightarrow{H} Y)$
$\uparrow C_{n+1}(Y)$

[Check: This is indeed a chain htpy, equivalently
$$\underbrace{\partial P}_{\substack{\text{full bdry of} \\ \text{prism } \partial(\Delta^n \times I)}} = \underbrace{C_\circ(g)}_{\substack{\text{top face} \\ \Delta^n \times \{1\}}} - \underbrace{C_\circ(f)}_{\substack{\text{bottom face} \\ \Delta^n \times \{0\}}} - \underbrace{P\partial}_{\substack{\text{lateral faces} \\ \partial\Delta^n \times I}}$$
] ∎

Long exact sequences & excision:

Q: how does homology interact with quotient of topological spaces?

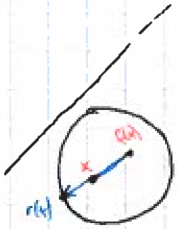
Thm: If $A \overset{j}{\hookrightarrow} X \xrightarrow{q} X/A$ is a good subspace inclusion, then $\exists$ a long exact sequence:

$$\begin{tikzcd} \dots \arrow[r, "\partial_{n+1}"] & \tilde H_n(A) \arrow[r, "\tilde H_n(j)"] & \tilde H_n(X) \arrow[r, "\tilde H_n(q)"] & \tilde H_n(X/A) \arrow[dll, "\partial_n"'] \\ & \tilde H_{n-1}(A) \arrow[r, "\tilde H_{n-1}(j)"] & \dots & \\ \dots \arrow[r, "\tilde H_0(j)"] & \tilde H_0(X) \arrow[r, "\tilde H_0(q)"] & \tilde H_0(X/A) \arrow[r] & 0. \end{tikzcd}$$

"boundary homomorphism"

Cor: $\forall\ n \ge 0$, $\tilde H_i(S^n) \cong \begin{cases} \mathbb{Z}, & i = n \\ 0, & i \ne n \end{cases}$

(proved in L7)



Pf: Previously proved for n=0.

$C_\circ(S^0) = C_\circ(pt) \oplus C_\circ(pt)$, so

$\tilde C_\circ(S^0) = (\dots \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\binom{0\ 0}{0\ 0}} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\varepsilon} \mathbb{Z})$ ε = sum of coeff's $= (1\ 1)$

$\rightsquigarrow \tilde H_0(S^0) = \mathbb{Z}\{(1,-1)\}$.

For $n > 0$, $S^{n-1} \hookrightarrow D^n$ is a good subspace incl^n.
Also, D^n is contractible, so $\tilde H_*(D^n) \cong \tilde H_*(pt) = 0$
i.e. $D^n \to pt$ is a hmtpy equiv

So by LES, we get ES's $D^n/S^{n-1} \cong S^n$

$$0 = \tilde H_i(D^n) \longrightarrow \tilde H_i(S^n) \xrightarrow{\partial_i} \tilde H_{i-1}(S^{n-1}) \to \tilde H_{i-1}(D^n) = 0$$

i.e. ∂_i is an iso. im = 0 $\Rightarrow$ inj   surj $\Leftarrow$ ker: everything

i.e. $\tilde H_i(S^n) \cong \tilde H_{i-1}(S^{n-1})$ $\forall\ i$.

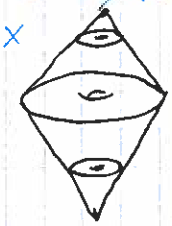
So, claim follows by induction. ∎

More generally, suspension iso:

$$X \overset{j}{\hookrightarrow} \underbrace{CX}_{\text{contractible}} := X \times I / X \times \{0\}$$

$\hookrightarrow \Sigma X := CX / X = (X \times I / X \times \{0\}) / X \times \{1\}$
the suspension of X

$\tilde H_i(\Sigma X) \cong \tilde H_{i-1}(X)$



Cor: (Brouwer's fixed pt thm): $\nexists$ a retraction of $S^{n-1} \overset{j}{\hookrightarrow} D^n$. Hence, every cts. fxn $D^n \xrightarrow{f} D^n$ has a fixed point.

Pf: A retraction would be a cts fcn $D^n \xrightarrow{r} S^{n-1}$ s.t. $r \circ j = id_{S^{n-1}}$

equivalently:

$$\begin{tikzcd} S^{n-1} \arrow[r, hook] \arrow[dr, "id_{S^{n-1}}"'] & D^n \arrow[d, dashed, "r"] \\ & S^{n-1} \end{tikzcd}$$

On $\tilde H_{n-1}$, this would give

$$\begin{tikzcd} \mathbb{Z} = \tilde H_{n-1}(S^{n-1}) \arrow[r, "\tilde H_{n-1}(j)"] \arrow[dr, "\tilde H_{n-1}(id_{S^{n-1}})"'] & \tilde H_{n-1}(D^n) = 0 \arrow[d, "\tilde H_{n-1}(r)"] \\ & \tilde H_{n-1}(S^{n-1}) = \mathbb{Z} \end{tikzcd}$$

$\tilde H_{n-1}(id_{S^{n-1}}) = id_{\tilde H_{n-1}(S^{n-1})} = id_{\mathbb{Z}}$

a contradiction!

Then, if a cts fxn $D^n \xrightarrow{f} D^n$ had no fixed pts, we could define a retraction onto $S^{n-1} = \partial D^n$ by:

Def: A <u>homotopy equivalence</u> is $X \xrightarrow{f} Y$ s.t. $\exists$ $X \xleftarrow{g} Y$ and homotopies $id_X \sim gf$ and $id_Y \sim fg$.

(An <u>iso</u> in any category is $X \xrightarrow{f} Y$ s.t. $\exists$ $X \xleftarrow{g} Y$ s.t. $id_X = gf$ and $id_Y = fg$)

Obs: If $\mathrm{Top} \xrightarrow{F} \mathcal{C}$ is hmtpy invariant

(i.e.

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow H"] & Y \end{tikzcd} \text{ in Top} \rightsquigarrow \begin{tikzcd} F(X) \arrow[r, bend left, "F(f)"] \arrow[r, bend right, "F(g)"'] \arrow[r, phantom, "\circlearrowleft"] & F(Y) \end{tikzcd} \text{ in } \mathcal{C})$$

then F carries htpy equiv$^{\text{ce}}$ to isos.

Def: For $A \overset{j}{\hookrightarrow} X$ a subspace, a <u>retraction</u> is $A \xleftarrow{r} X$ s.t. $r \circ j = id_A$; it's a <u>deformation retraction</u> if it's moreover "a homotopy equivalence relative to A" i.e. moreover $j \circ r \sim id_X$ rel A.

$H(x,0) = x$

$H(x,1) = jr$

$H(a,t) = a$

$\forall t \in I$ $\forall a \in A \subset X$

Def: A <u>pair</u> is $A \subseteq X \in \mathrm{Top}$. It's a good pair if $A \neq \emptyset$ and $\exists$ open $U \supset A$ s.t. $A \hookrightarrow U$ admits a deformation retraction.

The thm holds even for $A = \emptyset$ if we set $X/\emptyset = X$.

L7: Proof outline (of good subspace inclusion, LES...):

0) Define <u>relative homology</u> by $C_\bullet(X,A) := \frac{C_\bullet(X)}{C_\bullet(A)}$
$\rightsquigarrow H_n(X,A) := H_n(C_\bullet(X,A))$.

So by def^n, we have a (short) exact sequence in Ch:

$$\begin{tikzcd} 0 \arrow[r] & C_\bullet(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & C_\bullet(X) \arrow[r] \arrow[d, "C_\bullet(q)"] & C_\bullet(X,A) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C_\bullet(A/A) \arrow[r] & C_\bullet(X/A) \arrow[r] & C_\bullet(X/A, A/A) \arrow[r] & 0 \end{tikzcd}$$

1) Prove that $\longrightarrow$ is a quasi-isomorphism (iso on H_n $\forall n \in \mathbb{Z}$)

2) Prove that $\forall$ $y \in Y$ (e.g. X/A, A/A),
$\tilde{H}_*(Y) \xrightarrow{\cong} H_*(Y, \{y\})$

3) Prove $\forall$ SES $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ of ch cxes, get LES in H_*:

$$\begin{tikzcd} \cdots \arrow[r, "\partial_{n+1}"] & H_n(A_\bullet) \arrow[r] & H_n(B_\bullet) \arrow[r] & H_n(C_\bullet) \arrow[dll, "\partial_n"'] \\ & H_{n-1}(A_\bullet) \arrow[r] & \cdots \end{tikzcd}$$

Order of operations: 0, 3, 2, 1:

0) Whereas $H_n(X) := \frac{Z_n(C_\bullet(X))}{B_n(C_\bullet(X))}$

$$= \frac{\{\text{bdryless (singular) n-chains}\}}{\{\text{"trivial" ones}\}}$$

= "n-dimensional holes in X",

$$H_n(X,A) := \frac{Z_n(C_\bullet(X,A))}{B_n(C_\bullet(X,A))} := \frac{Z_n(C_\bullet(X)/C_\bullet(A))}{B_n(C_\bullet(X)/C_\bullet(A))}$$

$$= \frac{\{\text{equivalence classes of n-chains in X whose boundary lies in A}\}}{\{\text{"trivial" ones}\}}$$

carefully add elements of $C_n(A) \subseteq C_n(X)$

($\exists$ nontrivial elt of $H_n(X,A)$)

$$\approx \frac{\{\text{bdryless n-chains in } X/A\}}{\{\text{"trivial" ones}\}}$$

3) For SES in Ch $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$

<u>Claim:</u> get LES in hlgy (chain on lower left)

$$\begin{tikzcd} 0 \arrow[r] & A_{n+1} \arrow[r, hook] \arrow[d, "\partial"'] & B_{n+1} \arrow[r, two heads] \arrow[d, "\partial"'] & C_{n+1} \arrow[r] \arrow[d, "\partial"'] & 0 \\ 0 \arrow[r] & A_n \arrow[r, hook] \arrow[d, "\partial"'] & B_n \arrow[r, two heads] \arrow[d, "\partial"'] & C_n \arrow[r] \arrow[d, "\partial"'] & 0 \\ 0 \arrow[r] & A_{n-1} \arrow[r, hook] & B_{n-1} \arrow[r, two heads] & C_{n-1} \arrow[r] & 0 \end{tikzcd}$$

exactness @ $H_n(B_\bullet)$ i.e. $\mathrm{im}(H_n(f_\bullet)) = \ker(H_n(g_\bullet))$

$\subseteq$: For $[a] \in H_n(A_\bullet)$ represented by $a \in Z_n(A_\bullet) \subseteq A_n$

$H_n(g_\bullet) \circ H_n(f_\bullet)([a]) := H_n(g_\bullet)([f_n(a)])$
$:= [g_n(f_n(a))] = [0] = 0$

$\supseteq$: Say $[b] \in H_n(B_\bullet)$ is represented by $b \in Z_n(B_\bullet) \subseteq B_n$. Suppose $0 = H_n(g_\bullet)([b]) := [g_n(b)]$

Then, $\exists$ $c \in C_{n+1}$ s.t. $g_n(b) = \partial c$

Since g_{n+1} is surj., $\exists$ $\tilde{b} \in B_{n+1}$ s.t. $g_{n+1}(\tilde{b}) = c$. Then, $(b - \partial\tilde{b}) \in Z_n(B_\bullet)$ with $[b] = [b - \partial\tilde{b}]$ (same hlgy class)

On the other hand, $g(\partial\tilde{b}) = \partial g\tilde{b} = \partial\tilde{c} = gb$,

so $g(b - \partial\tilde{b}) = g(b) - g(\partial\tilde{b}) = g(b) - g(b) = 0$

So $\exists$ $a \in A_n$ s.t. $f_n(a) = b - \partial\tilde{b}$.

Moreover $\partial a = 0$ (i.e. $a \in Z_n(A_\bullet) \subseteq A_n$) because

$$f_{n-1}(\partial a) = \partial f_n a = \partial(b - \partial\tilde{b}) = \partial b - \underbrace{\partial\partial\tilde{b}}_{} = 0$$

0 b/c $b \in Z_n(B_\bullet)$; $= 0$ b/c $\partial^2 = 0$

and f_{n-1} is injective.

So, $[b] = [b - \partial\tilde{b}] = [f_n(a)] =: H_n(f_\bullet)([a])$

Def of $\partial_n : H_n(C_\cdot) \to H_{n-1}(A_\cdot)$.
(Exercise: check well defined, and exactness $\in H_n(C_\cdot)$ and $H_n(A_\cdot)$)

Given $[c] \in H_n(C_\cdot)$, choose $c \in Z_n(C_\cdot) \subseteq C_n$
Choose $b \in B_n$ s.t. $g_n(b) = c$, Consider $\partial b \in B_{n-1}$. This has $g_{n-1}(\partial b) = \partial g_n(b) = \partial c = 0$.
So $\exists!$ $a \in A_{n-1}$ s.t. $f_{n-1}(a) = \partial b$
★ We set $\partial_n([c]) := [a] \in H_{n-1}(A_\cdot)$
For $H_n(X,A) \xrightarrow{\partial_n} H_{n-1}(A)$, this takes an n-chain in X w/ boundary in A to its boundary! (recognizing as lying in $A \subseteq X$).

L8: $H_n(X,A) \xrightarrow{\partial_n} H_{n-1}(A)$

an n-cycle in X w/ boundary in A $\longrightarrow$ its boundary

$$\begin{tikzcd} X \arrow[r] & Y \\ A \arrow[u, hook] \arrow[r, dashed] & B \arrow[u, hook] \end{tikzcd}$$

Remarks on rel hlgy:
Clearly, a map of pairs $(X,A) \to (Y,B)$ induces homomorphisms $H_n(X,A) \to H_n(Y,B)$.
Proposition: If $f, g : (X,A) \to (Y,B)$ are homotopic through maps of pairs, then $H_n(f) = H_n(g) : H_n(X,A) \to H_n(Y,B)$.
Pf: By assumption, the chain hmtpy descends to the quotient

$$\begin{tikzcd} C_\cdot(X) \arrow[r, bend left, "C_\cdot(f)"] \arrow[r, bend right, "C_\cdot(g)"'] \arrow[d] & C_\cdot(Y) \arrow[d] \\ C_\cdot(X,A) & C_\cdot(Y,B) \end{tikzcd}$$

For later: A triple is $B \subseteq A \subseteq X$.
Prop: LES $\cdots \longrightarrow H_n(A,B) \to H_n(X,B) \to H_n(X,A) \to \cdots$
Pf: SES

$$\begin{tikzcd} 0 \arrow[r] & C_\cdot(A,B) \arrow[r] \arrow[d, equal] & C_\cdot(X,B) \arrow[r] \arrow[d, equal] & C_\cdot(X,A) \arrow[r] \arrow[d, equal] & 0 \\ & C_\cdot(A)/C_\cdot(B) & C_\cdot(X)/C_\cdot(B) & \frac{C_\cdot(X)}{C_\cdot(A)} \cong \frac{C_\cdot(X)/C_\cdot(B)}{C_\cdot(A)/C_\cdot(B)} & \end{tikzcd}$$

2) Observe

$$\begin{tikzcd} 0 \arrow[r] & C^{A}_\cdot(\{y\}) \arrow[r] \arrow[d, "\cong"'] & C_\cdot(Y) \arrow[r] \arrow[d, "\cong"'] & \tilde{C}_\cdot(Y) \arrow[r] \arrow[d, dashed, "\cong"'] & 0 \\ 0 \arrow[r] & C_\cdot(\{y\}) \arrow[r] & C_\cdot(Y) \arrow[r] & C_\cdot(Y,\{y\}) \arrow[r] & 0 \end{tikzcd}$$

$= \mathbb{Z}$ ($\sim$ means in dim 0) $\mathbb{Z} \oplus \cdots \to \mathbb{Z}$

$\cong$: quasi-iso

Proof: The five lemma: Given

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & D \arrow[r] \arrow[d, "\delta"'] & E \arrow[d, "\varepsilon"'] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \end{tikzcd}$$

with both rows exact, if $\alpha, \beta, \gamma, \delta$ are iso's, then so is γ.
Use that SES on ch cxes gives LES on hlgy, plus ⤴
Better: ① β, δ surj & ε inj $\Rightarrow \gamma$ surj.
② β, δ inj & α surj $\Rightarrow \gamma$ inj. Pf: Diagram chase!

1) Use another key property of homology called excision
Thm (excision, v1): Given $Z \subseteq A \subseteq X$ s.t. $\overline{Z} \subseteq \mathring{A}$, (i.e. $\overline{A} \subseteq \text{int} X$)
$(X \setminus Z, A \setminus Z) \hookrightarrow (X,A)$ induces an iso on rel hlgy.
Thm (excision, v2): Given $A, B \subseteq X$ s.t. $\mathring{A} \cup \mathring{B} = X$,
$(B, A \cap B) \hookrightarrow (X,A)$ induces an iso on rel hlgy.
Pf. modulo lemma:
Write $C_\cdot(A+B) := \underbrace{C_\cdot(A) + C_\cdot(B)}_{} \subseteq C_\cdot(X)$

$\leftrightarrow$ subgp gen^d by $C_n(A)$ & $C_n(B)$

Then, get

$$C_\cdot(B, A \cap B) := \frac{C_\cdot(B)}{C_\cdot(A \cap B)} \xrightarrow{\cong} \frac{C_\cdot(A+B)}{C_\cdot(A)} \xrightarrow{\overset{\text{by lemma}}{\simeq}} \frac{C_\cdot(X)}{C_\cdot(A)} =: C_\cdot(X,A)$$

In dim n, these are both $\mathbb{Z}\{\text{hom}_{Top}(\Delta^n, B) \setminus \text{hom}_{Top}(\Delta^n, A \cap B)$

L9: Claim 1: For $A \hookrightarrow X$ (for pf of LES), a good subspace inclusion, $H_*(X,A) \xrightarrow{\cong} H_*(X/A, A/A)$.

Closed, $\exists$ open '$A \hookrightarrow U \hookrightarrow X$' s.t. A a def. ret. of U

Main input:
Excision: v1: $Z \subseteq A \subseteq X$ s.t. $\overline{Z} \subseteq \mathring{A}$, $H_*(X \setminus Z, A \setminus Z) \xrightarrow{\cong} H_*(X,A)$
v2: $A, B \subseteq X$ s.t. $\mathring{A} \cup \mathring{B} = X$, $H_*(B, B \cap A) \xrightarrow{\cong} H_*(X,A)$
Pf of v2: $C_\cdot(A+B) = C_\cdot(A) + C_\cdot(B) \subseteq C_\cdot(X)$
Then, $\frac{C_\cdot(B)}{C_\cdot(B \cap A)} \xrightarrow{\cong} \frac{C_\cdot(A+B)}{C_\cdot(A)} \longrightarrow \frac{C_\cdot(X)}{C_\cdot(A)}$
Lemma: Say $\mathcal{U} = \{U_\alpha \subseteq X\}_{\alpha \in A}$ s.t. $\bigcup_{\alpha \in A} \mathring{U}_\alpha = X$
Write $C^{\mathcal{U}}_\cdot(X) := \left(\sum_{\alpha \in A} C_\cdot(U_\alpha)\right) \subseteq C_\cdot(X)$
Then, $C^{\mathcal{U}}_\cdot(X) \hookrightarrow C_\cdot(X)$ is a quasi iso.
Pf: Hw.

$$\begin{tikzcd} 0 \arrow[r] & C_\cdot(A) \arrow[r, hook] \arrow[d, equal] & C_\cdot(A+B) \arrow[r] \arrow[d, hook, "\text{q.i. by lemma}"] & \frac{C_\cdot(A+B)}{C_\cdot(A)} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C_\cdot(A) \arrow[r, hook] & C_\cdot(X) \arrow[r] & \frac{C_\cdot(X)}{C_\cdot(A)} \arrow[r] & 0 \end{tikzcd}$$

$\leftarrow$ 2 of 3 property for q.i. among SES of ch cx ($\leftarrow$ 5 lemma)

Pf of claim 1: (Prop 2.22) Choose $A \overset{\text{def ret}}{\hookrightarrow} U \overset{\text{open}}{\hookrightarrow} X$ as guaranteed by "good".

$$\begin{tikzcd} & & X \arrow[dd] & & X \setminus A \arrow[ll, hook'] \arrow[dd, "\cong"] \\ & U \arrow[ur, hook] \arrow[dd] & & U \setminus A \arrow[ll, hook'] \arrow[ur, hook] \arrow[dd, "\cong"] & \\ A \arrow[ur, hook, "\text{def ret}"] \arrow[dd] & & X/A & & (X/A) \setminus (A/A) \\ & U/A \arrow[ur, hook] & & (U/A) \setminus (A/A) \arrow[ll, hook'] \arrow[ur, hook] & \\ A/A \arrow[ur, hook, "\text{def ret}"] & & & & \end{tikzcd}$$

homeo

by excision v1
$Z \subseteq A$ is $A \subseteq U$
$A/A \subseteq U/A$

$$\begin{tikzcd} H_*(X,A) \arrow[r, "\cong"] \arrow[d] & H_*(X,U) \arrow[d] & H_*(X \setminus A, U \setminus A) \arrow[l, "\cong"'] \arrow[d] \\ H_*(X/A, A/A) \arrow[r, "\cong"'] & H_*(X/A, U/A) & H_*((X/A) \setminus (A/A), (U/A), (A/A) \arrow[l, "\cong"'] \end{tikzcd}$$

LES of triple $(A \subseteq U \subseteq X)$, b/c $H_*(U,A) = 0$
Similarly, $H_*(U/A, A/A) = 0$

hence iso!

■

Corr (Ex 2.23):

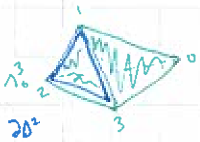
① $H_n(D^n, \partial D^n)$ is generated by $\Delta^n \xrightarrow{\text{any homeo}} D^n$

② $\tilde{H}_n(S^n) \overset{\text{canonically}}{\cong} \mathbb{Z}$, $S^n = \Delta^n_N \coprod_{\partial\Delta^n} \Delta^n_S$

Pf: ① Induction, $n=0$ trivial

union of all faces except 0th

$$\begin{tikzcd} \Lambda^n_0 \arrow[r, hook] & \partial\Delta^n \arrow[r, hook] & \Delta^n \\ \partial\Delta^{n-1} \arrow[u, hook] \arrow[r, hook] & \Delta^{n-1} \arrow[u, hook, "0^{\text{th}}\text{ face}"'] & \end{tikzcd}$$



by LES of triple $H_*(\Delta^n, \Lambda^n_0) = 0$

$$\begin{tikzcd} H_n(\Delta^n, \partial\Delta^n) \arrow[r, "\cong"] & H_{n-1}(\partial\Delta^n, \Lambda^n_0) \arrow[r] & H_{n-1}(\partial\Delta^n/\Lambda^n_0, \Lambda^n_0/\Lambda^n_0) \\ & H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}) \arrow[u, "\cong"] \arrow[r] & H_{n-1}(\Delta^{n-1}/\partial\Delta^{n-1}, \partial\Delta^{n-1}/\partial\Delta^{n-1}) \arrow[u, "\text{homeo of pairs}"'] \end{tikzcd}$$

$[id_{\Delta^n}]$ $\quad$ $[\sum_{i=0}^{n} (-1)^i \delta_i^n(id_{\Delta^n})]$

$\cong$ by claim ①, both good subspaces

in rel htpy! $=$ $[\delta_0^n(id_{\Delta^n})]$ $\quad$ $\in$ $[id_{\Delta^{n-1}}]$

Cor 2.25: For good basepoints $x_\alpha \subseteq X_\alpha$,

$$\bigoplus_\alpha \tilde{H}_n(X_\alpha) \xrightarrow{\cong} \tilde{H}_n\left(\bigvee_\alpha X_\alpha\right)$$

Pf: $H_n(\coprod_\alpha X_\alpha, \coprod_\alpha \{x_\alpha\}) \cong H_n(\underbrace{\bigvee_\alpha X_\alpha}_{=(\amalg X_\alpha)/(\amalg x_\alpha)}, *) \cong \tilde{H}_n(\bigvee_\alpha X_\alpha)$

$\quad = \bigoplus_\alpha H_n(X_\alpha, \{x_\alpha\}) = \bigoplus_\alpha \tilde{H}_n(X_\alpha)$

Cor: (invariance of dim): For nonempty open $U \in \mathbb{R}^m$, $V \subseteq \mathbb{R}^n$, if $\exists$ homeo $U \cong V$ then $m=n$.

Def: Then the local homology of $X \ni x$ is $H_n(X, X\setminus\{x\})$. An observation is that if x is closed, then $\forall$ $x \in U \subseteq X$,

$$H_n(X, X\setminus\{x\}) \xleftarrow{\cong} H_n(U, U\setminus\{x\})$$

(by excising $X\setminus U$)

Pf: Pick $x \in U$, open ball nbhd $x \in D \subset U$.

$$H_d^{loc}(U,x) \cong H_d^{loc}(D,x) \cong H_d^{loc}(\mathbb{R}^m, 0) := H_d(\mathbb{R}^m, \mathbb{R}^m\setminus\{0\})$$

$$\left.\begin{matrix} \mathbb{Z} & n=d \\ 0 & \text{o.w.} \end{matrix}\right\} = \tilde{H}_{d-1}(S^{n-1}) \cong H_{d-1}(\mathbb{R}^n\setminus\{0\}) \quad \downarrow\text{LES}$$

And homeo $U \xrightarrow{f} V$ gives

$$H_*^{loc}(U,x) \xrightarrow{\cong} H_*^{loc}(V, f(x))$$ ■

L 10: A little philosophy...

Observation: For any map $A \xrightarrow{f} B$, we can form the mapping cylinder:

$$Cyl(f) := (A\times I) \coprod_A B = \operatorname{colim}\begin{pmatrix} A \xrightarrow{f} B \\ \downarrow (id_A, const_0) \\ A\times I \end{pmatrix}$$

$(a,0) \mapsto a$ $\quad$ f

And we get:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "="'] \arrow[dr, phantom, "\checkmark"] & B \arrow[d, "\simeq"] \\ A \arrow[r, hook, "(id_A, const_0)"'] & Cyl(f) \end{tikzcd}$$

homotopy equivalence! (in fact, a deformation retraction: collapse the cylinder)

a good subspace inclusion

Commutes up to homotopy.

So in homology we get:

$$\begin{tikzcd} H_*(A) \arrow[r, "H_*(f)"] \arrow[d, "="'] \arrow[dr, phantom, "\circlearrowright"] & H_*(B) \arrow[d, "\cong"] \\ H_*(A) \arrow[r, "H_*(id_A, const_0)"'] & H_*(cyl(f)) \end{tikzcd}$$

hence, this fits into a LES for homology with the mapping cone of f:

$cone(f) = cyl(f)/A$

the homotopy quotient $B\overset{h}{/}A$

($\simeq B/A$ when f is a good subspace inclusion).

★ In Ch, we also have a notion of cyl, cone, and homotopy quotients. For an inclusion of ch cxes, ordinary quotient $\simeq$ htpy quotient

So, $\boxed{\text{Top} \xrightarrow{C_*} \text{Ch preserves homotopy quotients.}}$

$$\begin{tikzcd} C_\bullet(A) \arrow[r] \arrow[d, "="'] & C_\bullet(B) \arrow[r] \arrow[d, "\cong"] & C_\bullet(B)\overset{h}{/}C_\bullet(A) \arrow[d, "\cong"] & \\ 0 \arrow[r] & C_\bullet(A) \arrow[r, hook] & C_\bullet(cyl(f)) \arrow[r] & C_\bullet(cyl(f))/C_\bullet(A) \arrow[r] & 0 \end{tikzcd}$$

$$\cong$$

$$C_\bullet(cyl(f)/A) =: C_\bullet(cone(f)).$$

Def: An n-cell attachment to a space X' is

$$\begin{tikzcd} S^{n-1} \arrow[r, "f"] \arrow[d, hook] & X' \arrow[d] \\ cone(S^{n-1}) = D^n \arrow[r] & X := X' \coprod_{S^{n-1}}^{f} D^n \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

the colimit gluing $= cone(f)$ $\quad$ standard inclusion

⇝ get LES in H_* for

$$\begin{tikzcd} S^{n-1} \arrow[r, "f"] \arrow[d] & X' \arrow[d, "\simeq"] & \\ S^{n-1} \arrow[r, hook] & cyl(f) \arrow[r] & cone(f) =: X \end{tikzcd}$$

This suggests that H_* is completely determined by $H_*(pt)$ ([illegible] + LES ⇝ $H_*(S^n)$...) for all cell complexes. (§2.3 Eilenberg-Steenrod axioms for H_*)

L11: $H^{\Delta}_* \cong H^{sing}_*$

Thm: X a Δ-cx, $A \subseteq X$ a sub Δ-cx, then

$$\begin{tikzcd}
H^{\Delta}_*(X,A) \arrow[r, "\cong"] \arrow[d, equal] & H^{sing}_*(X,A) \arrow[d, equal] \\
H_*\left(\frac{C^{\Delta}_*(X)}{C^{\Delta}_*(A)}\right) & H_*\left(\frac{C^{sing}_*(X)}{C^{sing}_*(A)}\right)
\end{tikzcd}$$

$$\begin{tikzcd}
C^{\Delta}_n(X) \arrow[r] & C^{sing}_n(X)
\end{tikzcd}$$

$= \mathbb{Z}\{\underline{X_n}\}$ $\qquad$ $= \mathbb{Z}\{\hom_{Top}(\Delta^n, X)\}$

the set of n-simp's of the Δ-cx structure

$\sigma \longrightarrow$ it's "characteristic function" $\Delta^n \xrightarrow{\sigma} X$

Pf: First $A = \emptyset$ and X is finite dimensional. Write $X^k \subseteq X$ for the k-skeleton.

$$\begin{tikzcd}
0 \arrow[r] & C^{\Delta}_\bullet(X^{k-1}) \arrow[r, hook] \arrow[d, "①"'] & C^{\Delta}_\bullet(X^k) \arrow[r, two heads] \arrow[d, "③"] & C^{\Delta}_\bullet(X^k, X^{k-1}) \arrow[r] \arrow[d, "②"] & 0 \\
0 \arrow[r] & C^{sing}_\bullet(X^{k-1}) \arrow[r, hook] & C^{sing}_\bullet(X^k) \arrow[r, two heads] & C^{sing}_\bullet(X^k, X^{k-1}) \arrow[r] & 0
\end{tikzcd}$$

$k=0$:

$$\begin{tikzcd}
C^{\Delta}_\bullet(X^0) \arrow[r, "\approx"] \arrow[d, rightsquigarrow, "H_*"'] & C^{sing}_\bullet(X^0) \arrow[d] \\
H_i = \begin{cases} \mathbb{Z}\{X^0\}, & i=0 \\ 0 & , i\neq 0 \end{cases} \arrow[dr, "\cong"] & H^{sing}_i(X^0) \cong \bigoplus_{\sigma \in X^0} H^{sing}_i(\{\sigma\}) \\
 & = \begin{cases} \bigoplus_{X^0} \mathbb{Z}, & i=0 \\ 0 & , i\neq 0 \end{cases}
\end{tikzcd}$$

a point!

So, by induction, ① is a quasi-iso.

② is a quasi-iso, because...

$C^{\Delta}_\bullet(X^{k-1}) = (\mathbb{Z}\{X_{k-1}\} \to \dots \to \mathbb{Z}\{X_1\} \to \underline{\mathbb{Z}\{X_0\}})$

$C^{\Delta}_\bullet(X^k) = (\mathbb{Z}\{X_k\} \to \dots \to \mathbb{Z}\{X_1\} \to \underline{\mathbb{Z}\{X_0\}})$

So, $C^{\Delta}_i(X^k, X^{k-1}) \cong (\dots \xrightarrow{=} 0 \to \overset{\text{deg } k}{\overline{\mathbb{Z}\{X_k\}}} \to 0 \xrightarrow{=} 0 \xrightarrow{=} \dots)$

So, $H^{\Delta}_i(X^k, X^{k-1}) \cong \begin{cases} \mathbb{Z}\{X_k\} & i=k \\ 0 & i\neq k \end{cases}$

$$\begin{tikzcd}
\coprod_{X_k} \partial\Delta^k \arrow[r] \arrow[d, "good"'] & X^{k-1} \arrow[d, "good"] \\
\coprod_{X_k} \Delta^k \arrow[r] & X^k \arrow[ul, phantom, "\ulcorner" very near start, "pushout"]
\end{tikzcd}$$

i.e. $X^k = X^{k-1} \coprod_{\left(\coprod_{X_k} \partial\Delta^k\right)} \left(\coprod_{X_k} \Delta^k\right)$

So, $H^{sing}_i(X^k, X^{k-1})$

$\cong$ excision

$H^{sing}_i\left(\coprod_{X_k} \Delta^k, \coprod_{X_k} \partial\Delta^k\right)$

$\cong$

$\bigoplus_{X_k} H^{sing}_i(\Delta^k, \partial\Delta^k) \overset{\text{previously}}{=} \begin{cases} 0, & i\neq k \\ \mathbb{Z}\{[id_{\Delta^k}]\} & i=k \end{cases}$

③ Hence, also a quasi-iso by 2 out of 3 for SES of ch cxes ($\Leftarrow$ 5 lemma for ab gps)

For X possibly infinite dimensional:

key facts: ① Image of compact space under a continuous function is compact (And Δ^n is compact)

② A compact subspace of the Δ-cx X has nonempty ~~[illegible]~~ intersections with the interiors of only finitely many simplices.

So, every generator of $C^{sing}_n(X)$ lies in the image of $C^{sing}_n(X^k)$ for some k.

hence every element

In other words, $C^{sing}_\bullet(X) \cong \underline{\text{colim}}(C^{sing}_\bullet(X^0) \to C^{sing}_\bullet(X^1) \to \dots)$

increasing union

Hence, $H^{sing}_n(X) \cong \text{colim}_{k\to\infty}(H^{sing}_n(X^k))$

So by previous case, $H^{\Delta}_n(X) \xrightarrow{\cong} H^{sing}_n(X)$.

(Trivially, $C^{\Delta}_\bullet(X) \cong \text{colim}_{k\to\infty}(C^{\Delta}_\bullet(X^k))$ and $H^{\Delta}_\bullet(X) \cong \text{colim}_{k\to\infty}(H^{\Delta}_\bullet(X^k))$.)

For $A \neq \emptyset$,

$$\begin{tikzcd}
0 \arrow[r] & C^{\Delta}_\bullet(A) \arrow[r, hook] \arrow[d, "\cong"'] & C^{\Delta}_\bullet(X) \arrow[r, two heads] \arrow[d, "\cong"'] & C^{\Delta}_\bullet(X,A) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & C^{sing}_\bullet(A) \arrow[r, hook] & C^{sing}_\bullet(X) \arrow[r, two heads] & C^{sing}_\bullet(X,A) \arrow[r] & 0
\end{tikzcd}$$

by absolute case

(hence by 2of 3).

L12: Cellular Homology

Δ-cxes: Inductively gluing Δ^n's on along simplicial maps from $\partial\Delta^n$

CW-cxes: Inductively gluing D^n's on along <u>arbitrary</u> maps from $\partial D^n \cong S^{n-1}$

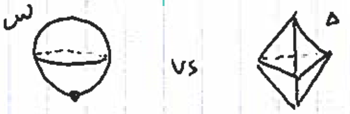
Ex:

$$\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d, hook] & pt \arrow[d] \\
D^n \arrow[r] & S^n \arrow[ul, phantom, "\ulcorner" very near start]
\end{tikzcd}$$

pushout = "gluing" in Top

a CW-cx str on S^n with one 0-cell and one n-cell

compare with $S^n \cong \Delta^n \cup_{\partial\Delta^n} \Delta^n$



Def: A <u>0-dim CW</u> cx is a set with discrete topology

Def: Given a space X', an <u>n-dimensional cell attachment</u> to X' is specified by a map $\partial D^n = S^{n-1} \xrightarrow{\alpha} X'$, and the result is

$$X := X' \underset{\alpha \nwarrow S^{n-1} \nearrow inc}{\bigcup} D^n = (X' \sqcup D^n) / \forall p \in S^{n-1}\ \alpha(p) \sim i(p)$$

with the quotient topology.

More generally, we can attach many cells at once:

given $\{S^{n_\beta - 1} \xrightarrow{\alpha_\beta} X'\}_{\beta \in B}$ ← indexing set, $n_\beta \in \mathbb{N}$

we get $X := X' \underset{\left(\coprod_{\beta\in B} S^{n_\beta - 1}\right)}{\bigcup} \left(\coprod_{\beta \in B} D^{n_\beta}\right)$

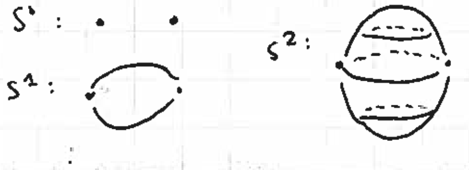
Def: An <u>n-dimensional CW-cx</u> is a space obtained by attaching n-cells to an $\underbrace{(n-1)\text{-dimensional CW-cx}}_{\text{the } n-1 \text{ skeleton of the CW-cx}}$.

(All attaching maps land in the n-1 skeleton)

CW-complexes are much more general and efficient than Δ-cxes (but H_*^{cell} slightly less algorithmic than H_*^{Δ}: requires the notion of the degree of a map $S^{n-1} \to S^{n-1}$).

Ex:
$$\begin{tikzcd}
S^{n-1}_N \sqcup S^{n-1}_S \arrow[r, "{(id,id)}"] \arrow[d] & S^{n-1} \arrow[d] \\
D^n_N \sqcup D^n_S \arrow[r] & S^n \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Inductively, a CW-str on S^n with two i-cells $\forall\ 0 \leq i \leq n$



⋮

S^∞: a CW-cx with two n-cells $\forall\ n \geq 0$

A <u>CW-cx</u> is an increasing union of n-dim CW-cxes. i.e. $X = \bigcup_{n \geq 0} X^{(n)}$ and $X^{(n-1)}$ is the $(n-1)$ skeleton of $X^{(n)}$.

↳ This has the advantage that the $\mathbb{Z}/2$ action by antipodes <u>is</u> via cellular maps.

Ex: $\mathbb{RP}^n := \{\text{lines}^{\text{through } \vec{0}} \text{ in } \mathbb{R}^{n+1}\} \cong S^n/p \sim -p\ \forall p \in S^n$

$\supset$ equatorial S^{n-1}, and $S^{n-1}/p \sim -p \cong \mathbb{R}P^{n-1}$

Complement ↗ is $(S^n \setminus S^{n-1})/p \sim -p$

$= (\mathring{D}^n_N \sqcup \mathring{D}^n_S)/p \sim -p \cong \mathring{D}^n$

So, $\mathbb{RP}^n \cong \mathbb{RP}^{n-1} \sqcup D^n$

$\overset{??}{\nwarrow}\ \partial D^n \nearrow \text{inc}$

$$\begin{tikzcd}
S^{n-1} \arrow[r, "\alpha"] \arrow[d] & X' \arrow[d] & & S^{n-1} \arrow[r, hook] \arrow[d] & \mathrm{cyl}(\alpha) \simeq X' \arrow[d] \\
D^n \arrow[r] \arrow[d] & X \arrow[ul, phantom, "\ulcorner", very near start] & {} \arrow[r, squiggly] & \mathrm{pt} \arrow[r] & X \arrow[ul, phantom, "\ulcorner", very near start] \\
\mathrm{cone}(S^{n-1}) & & & &
\end{tikzcd}$$

↙ homotopy

LES relating $\tilde{H}_*(S^{n-1})$, $\tilde{H}_*(X')$, & $\tilde{H}_*(X)$

↳ By induction, this gives a CW-str on $\mathbb{RP}^n$ with <u>one</u> i-cell $\forall\ i \leq n$ (likewise for $n = \infty$)

$(\mathbb{RP}^\infty = \bigcup_{n \geq 0} \mathbb{RP}^n)$

═

L13:

"Parameterization/atlas"-type definition:

Given $X \in \mathrm{Top}$, a CW-structure is: a set of characteristic maps $\{D^{n_\beta} \xrightarrow{\Phi_\beta} X\}_{\beta \in B}$ s.t.:

$\mathring{D}^{n_\beta}$

☆ each composite
$\mathring{D}^{n_\beta} \hookrightarrow D^{n_\beta} \xrightarrow{\Phi_\beta} X$ is a homeo onto its image, denoted $e_\beta^{n_\beta} \subseteq X$

☆ as a set, $X = \coprod_{\beta \in B} e_\beta^{n_\beta}$

☆ for every $\beta \in B$, the composite
$\partial D^{n_\beta} \hookrightarrow D^{n_\beta} \xrightarrow{\Phi_\beta} X$
lies in the union of cells of dimension $< n_\beta$.

☆ $\forall$ <u>subsets</u> $Z \subseteq X$ is closed iff $Z \cap e_\beta^{n_\beta} \subseteq e_\beta^{n_\beta}$ is closed $\forall\ \beta$.

Ex: $\mathbb{CP}^n := \{\mathbb{C}\text{-lines through } \vec{0} \text{ in } \mathbb{C}^{n+1}\}$

1-d comple w/o origi

$:= (\mathbb{C}^{n+1} \setminus \{0\}) / v \sim \lambda v\ \forall v \in \mathbb{C}^{n+1}, \forall \lambda \in \mathbb{C}^\times$

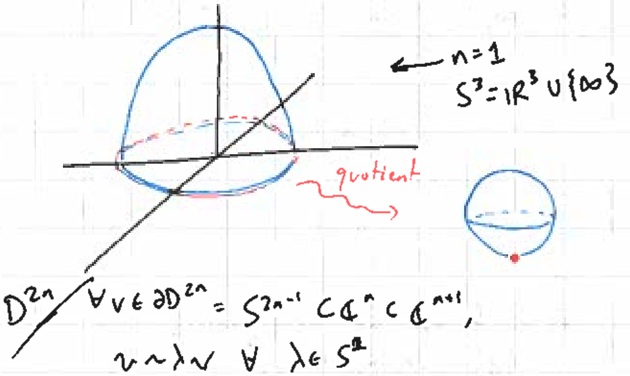
Aside: $(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^\times$ ← quotient by a grp action: if $G \curvearrowright X \in \mathrm{Set}$, $X/G := X/x \sim g \cdot x\ \forall x \in X, g \in G$

$\cong S^{2n+1}/v \sim \lambda v\ \forall v \in S^{2n+1} \subset \mathbb{C}^{n+1}\ \forall \lambda \in S^1 := U(1) \subset \mathbb{C}$

Observation: We can always rescale by $\lambda \in S^1$ so that the last coordinate of $v \in S^{2n+1} \subset \mathbb{C}^{n+1}$ is in $\mathbb{R}_{\geq 0} \subset \mathbb{C}$, and this is <u>unique</u> so long as that coordinate is nonzero.

$\cong \{v \in S^{2n+1} : v_n \in \mathbb{R}_{\geq 0}\} / \forall\ v = (v_0, \ldots, v_{n-1}, 0),\ v \sim \lambda v\ \forall \lambda \in S^1$.

$v = (v_0, \ldots, v_n)$



$\cong D^{2n} / \forall v \in \partial D^{2n} = S^{2n-1} \subset \mathbb{C}^n \subset \mathbb{C}^{n+1}$,

$v \sim \lambda v\ \forall\ \lambda \in S^1$

☆ So, $\mathbb{CP}^n \cong \mathbb{CP}^{n-1} \cup_{S^{2n-1}} D^{2n}$. (And $\mathbb{CP}^0 = \mathrm{pt.}$)

i.e. $\mathbb{CP}^n$ has one $2k$-cell $\forall\ 0 \leq k \leq n$

<u>Degree</u>: Fix some dimension $n > 0$. The <u>degree</u> of a map $S^n \xrightarrow{f} S^n$ is $\deg(f) \in \mathbb{Z}$ s.t. $H_n(S^n) \xrightarrow{H_n(f)} H_n(S^n)$ is equal to mult. by $\deg(f)$

free rank 1 abelian group

(There's a canonical iso

$$\begin{array}{ccc}
\deg(f) & \longleftrightarrow & H_n(f) \\
\rotatebox{-90}{$\in$} & & \rotatebox{-90}{$\in$} \\
\mathbb{Z} & \cong & \hom_{Ab}(H_n(S^n), H_n(S^n)) \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
1 & \longleftrightarrow & id_{H_n(S^n)}
\end{array}$$

).

L14: Degree (for cellular homology)

Basic properties:

a) $\deg(\mathrm{id}_{S^n}) = 1$

c) If $f \sim g$, then $\deg(f) = \deg(g)$

b) If f is not surjective, then it is nullhomotopic and so $\deg(f) = 0$

equivalently,

$$\begin{tikzcd}
S^n \arrow[r, "f"] \arrow[dr, dotted] & S^n \arrow[r, mapsto, bend left, "H_n"] & H_n(S^n) \arrow[r, "H_n(f)"] \arrow[dr, dotted] & H_n(S^n) \\
 & S^n \setminus \{pt\} \cong \mathbb{R}^n \cong pt \arrow[u, hook] & & 0 \arrow[u]
\end{tikzcd}$$

d) For $S^n \xrightarrow{f} S^n \xrightarrow{g} S^n$, $\deg(g \circ f) = \deg(g) \cdot \deg(f)$

In particular, $(c + d) \Rightarrow \deg(\text{homotopy equivalence}) = \pm 1$

e) $\deg(\text{reflection}) = -1$

use explicit generator:

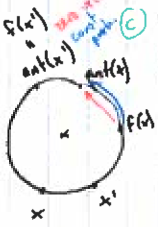
$\hat{\Delta^n_N} - \hat{\Delta^n_S} \mapsto \hat{\Delta^n_S} - \hat{\Delta^n_N}$ (reflection)

f) $\deg(\underbrace{\text{antipodal map}}) = (-1)^{n+1}$

composition of the $n+1$ reflections through coordinate hyperplanes

$$\begin{tikzcd}
S^n \arrow[d, "\text{antip.}"] \arrow[r, phantom, "\subset"] & \mathbb{R}^{n+1} \arrow[d] & (x_0, \dots, x_n) \arrow[r, mapsto] \arrow[d, mapsto] & (-x_0, x_1, \dots, x_n) \arrow[d] \\
S^n \arrow[r, phantom, "\subset"] & \mathbb{R}^{n+1} & (-x_0, \dots, -x_n) & (-x_0, -x_1, \dots, x_n) \arrow[l, mapsto]
\end{tikzcd}$$

g) If f has no fixed points, then $\deg(f) \overset{©}{=} \deg(\text{antipodal map}) \overset{f}{=} (-1)^{n+1}$



© from a htpy $\xrightarrow{f \sim \text{antip}:}$

If $f(x) \neq x$, then $\overrightarrow{f(x), \text{ant}(x)}$ does not pass through $\vec{0} \in \mathbb{R}^{n+1}$; the htpy $S^n \times I \to S^n$ carries $x \in S^n$ along the projection of this vector from $\vec{0}$ out to S^n at constant speed.

Thm 2.28: $\exists$ nonvanishing (continuous) vector field on S^n iff n is odd.

$$S^n = \{\vec{x} \in \mathbb{R}^{n+1} : |\vec{x}| = 1\} \leftarrow TS^n = \{(\vec{x}, \vec{v}) \in S^n \times \mathbb{R}^{n+1} : \vec{x} \cdot \vec{v} = 0\}$$

A $\overbrace{\text{fiber over}}^{\text{preimage}}$ $\vec{x} \in S^n$ is $T_{\vec{x}}S^n = \{\vec{v} \in \mathbb{R}^{n+1} : \vec{v} \perp \vec{x}\}$

Q: Is TS^n a trivial vector bundle?

$$\begin{tikzcd}
TS^n \arrow[rr, leftrightarrow, "\cong"] \arrow[dr] & & S^n \times \mathbb{R}^n \arrow[dl] \\
 & S^n &
\end{tikzcd}$$

?

Two vector bundles over S^1:

$$\begin{tikzcd}
S^1 \times \mathbb{R}^1 \arrow[d] & M \arrow[d] \\
S^1 & S^1
\end{tikzcd}$$

A section is a continuous function s such that $p \circ s = \mathrm{id}_B$

must be unique in the graph (no double loops in M)

$$\begin{tikzcd}
s(b) \in E_b \subset E \arrow[r, "p"] & B \ni b \arrow[l, dotted, bend left, "s"]
\end{tikzcd}$$

Fact: The trivial bundle $B \times \mathbb{R}^n \to B$ always has a nonvanishing section, $s(b) = (b, \vec{e}_i)$.

In fact, it has n fiberwise linearly independent sections, $\{(b, \vec{e}_i)\}_{1 \le i \le n}$.

Conversely, given a rk-n bdl $E \to B$, a set $\{s_i\}_{1 \le i \le n}$ of n fiberwise L.I. sections is a trivialization

$$\begin{tikzcd}
E \arrow[dr] & B \times \mathbb{R}^n \arrow[l, "\cong"'] \arrow[d] \\
 & B
\end{tikzcd}
\qquad
\begin{tikzcd}
\sum_{i=1}^n a_i s_i(b) \arrow[dr, mapsto] & (b, \sum_{i=1}^n a_i \cdot \vec{e}_i) \arrow[l, mapsto] \arrow[d, mapsto] \\
 & b
\end{tikzcd}$$

$$\underbrace{\left[\begin{matrix} TS^n \\ \downarrow \\ S^n \end{matrix} \text{ is trivial}\right] \Leftrightarrow \left[\begin{matrix} \exists\, n \text{ fiberwise LI} \\ \text{sections of } TS^n \end{matrix}\right]}_{\substack{\Uparrow \\ \text{Thm (1515)} \quad n = 2^k - 1}} \Rightarrow \left[\begin{matrix} \exists \text{ nonvanishing} \\ \text{section of } TS^n \end{matrix}\right]$$

$\Updownarrow$ n is odd

$n = 1, 3, 7$

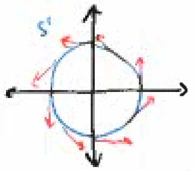
$\updownarrow \; \updownarrow \; \updownarrow$

$\mathbb{R}, \mathbb{C}, \mathbb{H}$

L15: Thm: $\exists$ a nonvanishing vector field (i.e. a nonvanishing section of $TS^n \to S^n$) on S^n iff n is odd.

Pf: For $n = 2K-1$ odd, define

$$v(x_1, \dots, x_{2k}) := (-x_2, x_1, \dots, -x_{2k}, x_{2k-1})$$



Conversely, suppose $\exists$ a nonvanishing section v of TS^n. Then $\frac{v}{|v|}$ is also a nonvanishing section of TS^n, i.e. WLOG, we can assume $|v| = 1$.

Now, we use v to get a htpy $id_{S^n} \Rightarrow$ antipodal map. $H: S^n \times I \to S^n$ given by

$$H(x,t) := \cos(\pi t)\cdot x + \sin(\pi t)\cdot v(x)$$

So, $\deg(ant) = \deg(id_{S^n}) = 1$

But $\deg(ant) = (-1)^{n+1}$ from before. So, n must be odd. ■

Proposition 2.29: If a group $G \curvearrowright S^{2n}$ freely, (i.e. any non identity element has no fixed pts), then $G = \mathbb{Z}/2$, or $G = \{e\}$.

Pf: Recall property g) of degree: if $S^k \xrightarrow{f} S^k$ has no fixed points, then $\deg(f) = \deg(\text{antip}) = (-1)^{k+1}$.

Action gives hom.

$$\begin{tikzcd}
G \arrow[r, "\alpha"] & \mathrm{Homeo}(S^n) \arrow[r, dashed, "\deg"] \arrow[d, equal] & \{\pm 1\} \arrow[dd, hook] \\
& \mathrm{aut}_{Top}(S^n) \arrow[d, hook] & \\
& \mathrm{end}_{Top}(S^n) \arrow[r, "\deg"] & \mathbb{Z}
\end{tikzcd}$$

(monoid hom)

By g), $\forall\ g \neq e \in G$, $\deg(\alpha(g)) = -1$. So, $\ker(\deg \circ \alpha) = \{e\}$. So $\deg \circ \alpha$ is inj ■

Local degree: For computing $\deg(S^n \xrightarrow{f} S^n)$, suppose $\exists\ y \in S^n$ s.t. $f^{-1}(y)$ is finite, say $f^{-1}(y) = \{x_1, \dots, x_m\}$. We define the local degree of f at x_i, and prove that $\deg(f) = \sum_{i=1}^m \deg(f|_{x_i})$.

Recall: $H_k^{loc}(z \in Z) := H_k(Z, Z \setminus \{z\})$

Want to define $\deg(f|x_i)$ to be

$$H_n^{loc}(x_i \in S^n) \dashrightarrow H_n^{loc}(y \in S^n) \cdots$$

but there's no such map of pairs!

$$\begin{tikzcd}
S^n \arrow[r, "f"] & S^n \\
S^n \setminus \{x\} \arrow[u, hook] \arrow[r, dashed, "\nexists"'] & S^n \setminus \{y\} \arrow[u, hook]
\end{tikzcd}$$

(if you remove y, you must remove all the preimages!)

Rather, choose $U_i \overset{open}{\subseteq} S^n$ s.t. $x_i \in U_i$, and $x_j \notin U_i\ \forall\ j \neq i$. Then,

$$\begin{tikzcd}
H_n^{loc}(x_i \in S^n) & & H_n^{loc}(y \in S^n) \\
& H_n^{loc}(x_i \in U_i) \arrow[ul, "\cong\ \text{excision}"'] \arrow[ur, "☺"] &
\end{tikzcd}$$

Other ingredient: use that $\forall\ z \in S^n$, $(S^n, \emptyset) \to (S^n, S^n \setminus \{z\})$ is an iso on H_n by LES.

$$\begin{tikzcd}
0 \arrow[r, "\cong"] & C_\cdot(\emptyset) \arrow[r] \arrow[d] & C_\cdot(S^n) \arrow[r, "\cong"] \arrow[d, equal] & C_\cdot(S^n, \emptyset) \arrow[r] \arrow[d, "\text{an iso on } H_n."] & 0 \\
0 \arrow[r] & \underbrace{C_\cdot(S^n \setminus \{z\})}_{H_* \text{ only in dim } 0} \arrow[r] & C_\cdot(S^n) \arrow[r] & C_\cdot(S^n, S^n \setminus \{z\}) \arrow[r] & 0
\end{tikzcd}$$

So, finally we get

$$\begin{tikzcd}
H_n(S^n) \arrow[rr, dashed, "\deg(f|_{x_i}) \in \mathbb{Z}"] \arrow[d, "\cong"'] & & H_n(S^n) \arrow[d, "\cong"] \\
H_n^{loc}(x_i \in S^n) & H_n^{loc}(x_i \in U_i) \arrow[l, "\cong"'] \arrow[r] & H_n^{loc}(y \in S^n)
\end{tikzcd}$$

Prop: $\deg(f) = \sum_{i=1}^m \deg(f|x_i)$.

Pf: $\forall\ 1 \leq i \leq m$

$$\begin{tikzcd}
& (S^n, \emptyset) \arrow[r, "f"] \arrow[d, "j"] \arrow[dl, "H_n\text{-iso}"'] & (S^n, \emptyset) \arrow[d, "H_n\text{-iso}"] \\
(S^n, S^n \setminus \{x_i\}) & \boxed{(S^n, S^n \setminus \{x_1, \dots, x_m\})} \arrow[l, "p_i"'] \arrow[r] & (S^n, S^n \setminus \{y\}) \\
& (U_i, U_i \setminus \{x_i\}) \arrow[u, "K_i"] \arrow[ul, "H_n\text{-iso}"] \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
H_n(S^n, S^n \setminus \{x_1, \dots, x_m\}) & H_n\left(\coprod_{i=1}^m U_i, \coprod_{i=1}^m U_i \setminus \{x_i\}\right) \arrow[l, "\overset{\text{excision}}{\cong}"'] \\
& \overset{\text{additivity}}{\cong} \bigoplus_{i=1}^m \underbrace{H_n(U_i, U_i \setminus \{x_i\})}_{H_n^{loc}(x_i \in U_i) \cong \mathbb{Z}} \arrow[ul, bend left, "H_n(K_i)"]
\end{tikzcd}$$

Observe: For $j \neq i$ $H_n(p_j) \circ H_n(K_i) = 0$

i.e. on H_n,

$$\begin{tikzcd}
& \mathbb{Z} \arrow[r, dashed, "\deg(f)"] \arrow[d] \arrow[dl, "\cong"'] & \mathbb{Z} \arrow[d, "\cong"] \\
\mathbb{Z} & \mathbb{Z}^{\oplus m} \arrow[l, "proj_j"'] \arrow[r, "\sum_{i=1}^m \deg(f|_{x_i})"] & \mathbb{Z} \\
& \mathbb{Z} \arrow[u, "incl_j"] \arrow[ul, "\cong"] \arrow[ur, "\deg(f|_{x_i})"'] &
\end{tikzcd}$$

L17: Lemma (for cellular homology): X a CW cx:

ⓐ $H_k(X^n, X^{n-1}) = \begin{cases} 0, & k \neq n \\ \mathbb{Z}\{\substack{n \text{ cells} \\ \text{of } X}\}, & k = n \end{cases}$

ⓑ $H_k(X^n) = 0$ for $k > n$

ⓒ $H_k(X^n) \to H_k(X)$ is $\begin{cases} \text{iso}, & k < n \\ \text{surj}, & k = n \end{cases}$

$$\begin{tikzcd} C_\bullet^{cell}(X) := (\cdots \arrow[r] & C_n^{cell}(X) \arrow[r, "d_n^{cell}"] \arrow[d, equal] & C_{n-1}^{cell}(X) \arrow[r] \arrow[d, equal] & \cdots) \\ \mathbb{Z}\{\substack{n \text{ cells} \\ \text{of } X}\} \overset{ⓐ}{\cong} & H_n(X^n, X^{n-1}) \arrow[dr, "\partial_n"'] & H_{n-1}(X^{n-1}, X^{n-2}) & \\ & & H_{n-1}(X^{n-1}) \arrow[u, "\text{canon.}"'] & \end{tikzcd}$$

$$\boxed{H_n^{cell}(X) := H_n(C_\bullet^{cell}(X))}$$

Thm:

$\partial^2 = 0$ & $H_n^{cell}(X) \cong H_n^{sing}(X)$

Pf: $\overset{"}{\ker(d_n^{cell})}/\operatorname{im}(d_n^{cell})$

$$\begin{tikzcd} & & & & H_n(X^{n+1}, X^n) \overset{ⓐ}{\cong} 0 & & \\ & 0 \overset{ⓑ}{\cong} H_n(X^{n-1}) \arrow[dr] & & H_n(X^{n+1}) \underset{ⓒ}{\cong} H_n(X) \arrow[ur] & & & \\ & & H_n(X^n) \arrow[ur] \arrow[dr, "\beta"] & & & & \\ \cdots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\alpha := \partial_{n+1}"] \arrow[rr, "d_{n+1}^{cell}"'] & & H_n(X^n, X^{n-1}) \arrow[rr, "d_n^{cell}"] \arrow[dr, "\gamma"'] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & \cdots \\ & \text{LES} & & & H_{n-1}(X^{n-1}) \arrow[ur] & \text{LES} & \\ & & & 0 \overset{ⓑ}{\cong} H_{n-1}(X^{n-2}) \arrow[ur] & & & \end{tikzcd}$$

First, $d_n^{cell} \circ d_{n+1}^{cell} = (\delta \circ \gamma) \circ (\beta \circ \alpha) = \delta \circ \overbrace{(\gamma \circ \beta)}^{0} \circ \alpha = 0.$

Then, $H_n(X) \overset{ⓒ}{\cong} H_n(X^{n+1}) \underset{\nearrow \text{exact}}{\cong} H_n(X^n)/\operatorname{im}(\alpha) \underset{\beta \text{ inj}}{\cong} \dfrac{\operatorname{im}(\beta)}{\operatorname{im}(\beta\alpha)} =: \operatorname{im}(\beta)/\operatorname{im}(d_{n+1}^{cell})$

$\overset{\searrow \text{exact}}{\cong} \dfrac{\ker(\gamma)}{\operatorname{im}(d_{n+1}^{cell})} \overset{\delta \text{ inj}}{\cong} \dfrac{\ker(\delta\gamma)}{\operatorname{im}(d_{n+1}^{cell})} = \dfrac{\ker(d_n^{cell})}{\operatorname{im}(d_{n+1}^{cell})} =: H_n^{cell}(X)$ ■

Remark: This is a very special case of a spectral sequence.
Namely, a spectral seq. for a filtration $F^n C_\bullet(X) := C_\bullet(X^n)$

Corollary: $H_n^{sing}(X)$ is generated by at most $\#\binom{n \text{ cells}}{\text{of } X}$ elements for any CW structure on X.
(e.g. $H_n = 0$ if $\nexists$ n-cells)

Corollary: If X has no $(n+1)$-cells or $(n-1)$cells, then

$$H_n(X) \cong \mathbb{Z}\{\substack{n\text{-cells} \\ \text{of } X}\}$$

Ex: $\mathbb{C}P^n = e^0 \cup e^2 \cup e^4 \cup \dots \cup e^{2n}$, so $H_k(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z} & 0 \leq k \leq 2n \text{ even} \\ 0 & o/w \end{cases}$

For X & Y CW cxes, $\exists$ CW str. on $X \times Y$ whose k-skeleton is

$$(X \times Y)^k = \bigcup_{i+j=k} X^i \times Y^j$$

Cells of $X \times Y$ are products of a cell of X and a cell of Y.

Ex: Recall $S^n = e^0 \cup e^n$.
We get a CW str on product:

$$S^n \times S^n = e^0 \cup e^n \cup e^n \cup e^{2n}$$

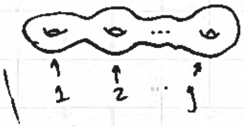
For $n > 1$, corrolary $\Rightarrow$ $H_k(S^n \times S^n) = \begin{cases} \mathbb{Z} & k = 0, 2n \\ \mathbb{Z} \oplus \mathbb{Z} & k = n \\ 0 & \text{o.w.} \end{cases}$

$$\begin{tikzcd} \text{Prop.} \; C_n^{cell}(X) \arrow[r, "d_n^{cell}"] \arrow[d, "\cong"] & C_{n-1}^{cell}(X) \arrow[d, "\cong"] \\ \mathbb{Z}\{\substack{n\text{-cells} \\ \text{of } X}\} \arrow[r, dashed] & \mathbb{Z}\{\substack{n-1 \text{ cells} \\ \text{of } X}\} \end{tikzcd}$$

$(\underset{\text{n-cell}}{\alpha}, \underset{\text{n-1 cell}}{\beta})$ matrix coefficients is the degree of

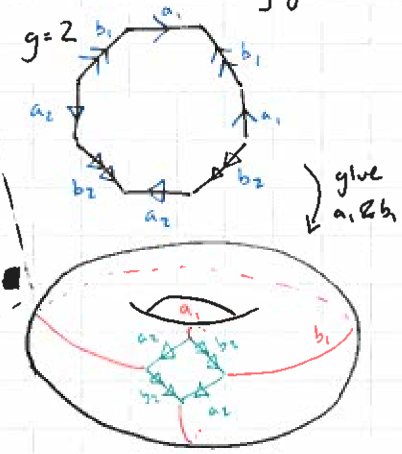
$$S^{n-1} \xrightarrow{\alpha^{th} \text{ attaching map}} X^{n-1} \xrightarrow{\text{quotient by complement of interior of } \beta^{th} \text{ cell}} S^{n-1}$$

Ex: $M = M_g$ the genus-g surface:



CW structure from 4g-gon:

$g = 2$



1 0-cell
2g 1-cells
1 2-cell

only possible attaching maps

attaching map: $S^1 \xrightarrow[{[a_1,b_1] \circ \dots \circ [a_g,b_g]}]{} M^1$

$[a,b] = a \circ b \circ a^{-1} \circ b^{-1}$

$$M^1 = \bigvee_{2g} S^1 = \left(\bigvee_{i=1}^{g} S^1_{a_j}\right) \vee \left(\bigvee_{j=1}^{g} S^1_{b_j}\right)$$

$$C_\bullet^{cell}(M_g) = (\mathbb{Z} \xrightarrow[\mathbf{0}]{} \mathbb{Z}^{\oplus 2g} \xrightarrow[0]{} \mathbb{Z})$$

in α^{th} summand degree of ⤳

$S^1 \xrightarrow[{[a,b_1] \circ \dots \circ [a_g,b_g]}]{} M^1 \xrightarrow{\text{quotient}} S^1_{a_i}$

$a_i \circ a_i^{-1}$ ⤳ has degree 0

all edges have both endpoints attached to the unique vertex

So,

$$H_k(M_g) \cong \begin{cases} \mathbb{Z} & k = 0, 2 \\ \mathbb{Z}^{\oplus 2g} & k = 1 \\ 0 & \text{o.w.} \end{cases}$$

null htpy

t=0

t=1/2

Cf: Hatcher · 3d examples

$T^3 = (S^1)^{\times 3}$ and $K^2 \times S^1$,

lens spaces $S^{2n-1} / (\mathbb{Z}/m)$

inside $U(1) = S^1 \subset \mathbb{C}^\times$, the m^{th} roots of x.

Ex: $\forall$ $G \in Ab$, $n \geq 1$, <u>Moore space</u> $M(G;n)$: $\tilde{H}_k = \begin{cases} G & k=n \\ 0 & \text{o.w.} \end{cases}$

Mayer Vietoris LES

$A, B \subseteq X$, $\mathring{A} \cup \mathring{B} = X$, $C_\cdot(A+B) \subseteq C_\cdot(X)$

⇝ SES

$$\begin{tikzcd}
0 \arrow[r] & C_\cdot(A\cap B) \arrow[r, hook, "\varphi", "{(j_A, -j_B)}"'] & C_\cdot(A)\oplus C_\cdot(B) \arrow[r, two heads, "\psi", "i_A + i_B"'] & C_\cdot(A+B) \arrow[r] \arrow[d, "\text{refinement lemma}", "\wr"'] & 0 \\
 & & & C_\cdot(X) &
\end{tikzcd}$$

⇝ LES in Homology $\cdots H_n(A+B) \xrightarrow{\partial_n} H_{n-1}(A\cap B) \to \cdots$

Variant: X a CW-cx, A, B sub-CW-cx, and $A \cup B = X$ (i.e. no need to pass to interiors).

∂_n: Geometric interpretation

$$\begin{tikzcd}
H_n(X) & H_n(A+B) \arrow[l, "\simeq"'] \arrow[r, "\partial_n"] & H_{n-1}(A\cap B) \\
 & Z_n(A+B) \arrow[u] & \\
 & z = x+y \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[r, mapsto, bend left] & \partial x \in Z_{n-1}(A\cap B) \arrow[uu]
\end{tikzcd}$$

Note: $0 = \partial z = \partial(x+y)$, so $\partial x = -\partial y$

$\Rightarrow \varphi(\partial x) = (\partial x, -\partial x) = (\partial x, \partial y)$

Ex of MV: $S^n = D^n \underset{S^{n-1}}{\cup} D^n$

$$\begin{tikzcd}
 & & & \tilde{C}_\cdot(S^n) & \\
\text{SES}: \quad 0 \arrow[r] & \tilde{C}_\cdot(S^{n-1}) \arrow[r, hook] & \underbrace{\tilde{C}_\cdot(D^n)\oplus\tilde{C}_\cdot(D^n)}_{=0} \arrow[r, two heads] & \tilde{C}_\cdot(D^n + D^n) \arrow[u, hook, "\wr"'] \arrow[r] & 0
\end{tikzcd}$$

0 on alternating terms Induces an iso on the LES in homology

So by induction $\tilde{H}_k(S^n) = \begin{cases} \mathbb{Z} & k=n \\ 0 & \text{o.w.} \end{cases}$

Cor: $\chi(X)$ where X has finite rank H_k,

$$\chi(X) = \chi(A) + \chi(B) - \chi(A\cap B)$$

E.g.

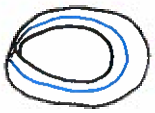
$\chi(S^0) = 2$

$\chi(S^1) = \chi(D^1) + \chi(D^1) - \chi(S^0) = 0$

$\rightsquigarrow \chi(S^n) = \chi(D^n) + \chi(D^n) - \chi(S^{n-1})$.

Ex: $K^2 = M \underset{S^1}{\cup} M$

klein bottle



$$\begin{tikzcd}
M \arrow[r, dotted, bend left, "r"] & S^1 \arrow[l, "\sim"'] \\
\partial M \cong S^1 \arrow[u, hook] \arrow[ru, "\text{degree } 2"'] &
\end{tikzcd}$$

$$0 = \tilde{H}_2(M)\oplus\tilde{H}_2(M) \to \tilde{H}_2(K) \xrightarrow{\partial} \tilde{H}_1(M\cap M) \to \tilde{H}_1(M)\oplus\tilde{H}_1(M) \to \tilde{H}_1(K) + \tilde{H}_0(M\cap M) = 0$$

$\tilde{H}_2(K) = 0$

$\tilde{H}_1(M\cap M) = \mathbb{Z} \longrightarrow \mathbb{Z}\oplus\mathbb{Z} = \tilde{H}_1(M)\oplus\tilde{H}_1(M)$

$1 \longmapsto (2, -2)$

$\tilde{H}_1(K) = \mathbb{Z}\oplus\mathbb{Z}/(2,-2)$

$\cong\downarrow$ change basis to $\{(1,0),(1,-1)\}$.

Homology with coefficients

G an ab. gp.

levelwise

$$\mathrm{Ch} \xrightarrow{(-)\otimes_{\mathbb{Z}} G} \mathrm{Ch}$$

$$C_\cdot(X; G) := C_\cdot(X) \otimes_{\mathbb{Z}} G$$

In dimension n,

$$C_n(X) \otimes_{\mathbb{Z}} G = \mathbb{Z}\{\hom_{Top}(\Delta^n, X)\} \otimes_{\mathbb{Z}} G \cong \underbrace{G^{\oplus \hom_{Top}(\Delta^n, X)}}$$

i.e. a $\hom_{Top}(\Delta^n, X)$-indexed direct sum of copies of G

All formulas and theorems go through without much change, but <u>computations</u> do change.

Notably: $H_k(pt; G) \cong \begin{cases} G & k=0 \\ 0 & k\neq 0 \end{cases}$ $\quad \tilde{H}_k(S^n; G) \cong \begin{cases} G & k=n \\ 0 & \text{o.w.} \end{cases}$

$H_k(S^0; G) \cong \begin{cases} G\oplus G & k=0 \\ 0 & k\neq 0 \end{cases}$

Lemma: If $S^k \xrightarrow{f} S^k$ has degree d, then $G \cong H_k(S^k; G) \longrightarrow H_k(S^k; G) \cong G$ is multiplication by d

Computing $H_k(K)$ with coefficients in $G = \mathbb{Z}/2$ yields:

$$\begin{tikzcd}
{} \arrow[r, "\cong"'] & \tilde{H}_1(M\cap M) \arrow[r] \arrow[d, equal] & \tilde{H}_1(M)\oplus\tilde{H}_1(M) \arrow[r, "\cong"'] \arrow[d, equal] & {} \\
 & \mathbb{Z}/2 \arrow[r, "\text{zero}"] & \mathbb{Z}/2\oplus\mathbb{Z}/2 & \\
 & 1 \arrow[r, mapsto] & (2\cdot 1, (-2)\cdot 1) = (0,0) &
\end{tikzcd}$$

So $\tilde{H}_k(K; \mathbb{Z}/2) = \tilde{H}_k(T^2; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & k=2 \\ \mathbb{Z}/2\oplus\mathbb{Z}/2 & k=1 \\ 0 & \text{o.w.} \end{cases}$

multiply by

$$\text{Ex: } C_\cdot^{cell}(\mathbb{R}P^n; \mathbb{Z}/2) = \left(\mathbb{Z}/2 \to \cdots \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{2} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2\right)$$

$$\shortparallel$$

$$C_\cdot^{cell}(\mathbb{R}P^n) \otimes_{\mathbb{Z}} \mathbb{Z}/2 = \left(\mathbb{Z}/2 \to \cdots \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2\right)$$

$\rightsquigarrow H_k(\mathbb{R}P^n; \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & 0 \leq k \leq n \\ 0 & \text{o.w.} \end{cases}$

Recall $H_k(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z} & 0 \leq k \leq 2n, \ k \text{ even} \\ 0 & \text{o.w.} \end{cases}$

$$\begin{tikzcd} C^{cell}_n(X) := H^{Sing}_n(X^n, X^{n-1}) \cong \mathbb{Z}\left\{\substack{n\text{-cells}\\ \text{of } X}\right\} \arrow[d, "d_n^{cell}"] \\ C^{cell}_{n-1}(X) \end{tikzcd}$$

d_n^{cell} = matrix of degrees of attaching maps

$$\substack{\alpha,\beta\\ \text{entry}} = \deg\left(\begin{tikzcd} D^n \\ S^{n-1} \arrow[u, hook] \arrow[r, "\alpha^{th}\text{ attaching map}"'] & X^{n-1} \arrow[r] & S^{n-1} \end{tikzcd}\right)$$

Thm: $H^{cell}_*(X) \cong H^{Sing}_*(X)$.

Ex: $\mathbb{RP}^n = e^0 \cup e^1 \cup \cdots \cup e^n$

↳ $C^{cell}_\bullet(\mathbb{RP}^n) = (\overset{n}{\mathbb{Z}} \to \cdots \to \overset{0}{\mathbb{Z}})$

For differentials, note that the attaching map for e^k is the double cover $S^{k-1} \xrightarrow{\varphi} \mathbb{RP}^{k-1} \xrightarrow{q} \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} \cong S^{k-1}$

The composite $q\varphi$ gives a homeo separately from each of ~~$S^{k-1}/\mathring{D}^{k-1}_{South}$~~ ~~and~~ ~~$S^{k-1}/\mathring{D}^{k-1}_{North}$~~

$(S^{k-1} \setminus \mathring{D}^{k-1}_{South})/\text{bdry}$ and $(S^{k-1} \setminus \mathring{D}^{k-1}_{North})/\text{bdry}$,

and these differ by precomposition by the antipodal map of S^{k-1}, which has degree $(-1)^k$. So, by the local degree formula ($\deg = \Sigma$ local degrees),

$$\begin{aligned} \deg(q\varphi) &= \deg(id) + \deg(antip) \\ &= 1 + (-1)^k = \begin{cases} 2, & k \text{ even} \\ 0, & k \text{ odd} \end{cases} \end{aligned}$$

i.e. $C^{cell}_\bullet(\mathbb{RP}^n) = (\mathbb{Z} \to \mathbb{Z} \xrightarrow{\cdots 2} \cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \underset{\sim}{\mathbb{Z}})$

So, $H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z}, & k=0 \\ \mathbb{Z}/2, & k \text{ odd and } 0<k<n \\ 0, & o/w \end{cases}$

Euler characteristic:

For X a finite CW cx, its Euler characteristic is:

$$\chi(X) = \sum_n (-1)^n \left(\substack{\#\text{ of } n\text{-cells}\\ \text{of } X}\right)$$

E.g. for X a triangulated surface,

$$\chi(X) = \#\text{ vertices} - \#\text{edges} + \#\text{ faces}$$

Thm: $\chi(X) = \sum_n (-1)^n \cdot \underbrace{rk(H_n(X))}_{\substack{\#\text{ of copies of } \mathbb{Z} \text{ that}\\ \text{show up in its primary decomposition.}}}$

i.e. It's a topological invariant! (independent of CW structure).

In fact, Thm ⇒ it's an invariant not just up to homeomorphism, but up to homotopy equivalence.

Prop: For any $C_\bullet \in Ch$ with all terms finitely generated and only finitely many nonzero terms,

$$\sum_{n\in\mathbb{Z}} (-1)^n \cdot rK(C_n) = \sum_{n\in\mathbb{Z}} (-1)^n \cdot rK(H_n(C_\bullet)) = \chi(C_\bullet)$$

Pf: Note the SES's

$$0 \to Z_n(C_\bullet) \hookrightarrow C_n \overset{\substack{C_{n-1}\\ \cup}}{\underset{}{\xrightarrow[\partial_n]{\text{image}}}} B_{n-1}(C_\bullet) \to 0$$

and

$$0 \to B_n(C_\bullet) \hookrightarrow Z_n(C_\bullet) \twoheadrightarrow H_n(C_\bullet) \to 0$$

Note: $rk(C_n) = rk(Z_n) + rk(B_{n-1})$

$rk(Z_n) = rk(B_n) + rk(H_n)$

So, $rk(C_n) = rk(H_n) + (rk(B_n) + rk(B_{n-1}))$.

Therefore,

$$\begin{aligned} \sum_{n\in\mathbb{Z}} (-1)^n rk(C_n) &= \sum_{n\in\mathbb{Z}} (-1)^n (rk(H_n) + (rk(B_n) + rk(B_{n-1}))) \\ &= \sum_{n\in\mathbb{Z}} (-1)^n rk(H_n) \end{aligned}$$

■

Pf of Thm:

$$\begin{aligned} \chi(X) &:= \sum_{n\in\mathbb{Z}} (-1)^n \left(\substack{\#\ n\text{-cells}\\ \text{in } X}\right) \\ &= \sum_{n\in\mathbb{Z}} (-1)^n \cdot rk(C^{cell}_n(X)) \\ &\overset{\text{Prop}}{=} \sum_{n\in\mathbb{Z}} (-1)^n \cdot rk(H^{cell}_n(X)) \\ &\overset{H^{cw}_* \cong H^{Sing}_*}{=} \sum_{n\in\mathbb{Z}} (-1)^n \cdot rk(H_n(X)) \end{aligned}$$

■

Aside: Not every pairs of triangulations share a common refinement.

Mayer-Vietoris LES (inclusion/exclusion principle)

For $A, B \subseteq X$ s.t. $X = \mathring{A} \cup \mathring{B}$, set $C_\bullet(A+B) \subseteq C_\bullet(X)$ as before. Recall refinement lemma: $C_\bullet(A+B) \xrightarrow{\simeq} C_\bullet(X)$ is a quasi-iso (i.e. iso on all H_n). Note we have the SES

$$0 \longrightarrow C_\bullet(A \cap B) \xhookrightarrow[\underbrace{(j_A, -j_B)}_{(C_\bullet(A\cap B \to A),\, -C_\bullet(A\cap B \to B))}]{} C_\bullet(A) \oplus C_\bullet(B) \xrightarrow[(i_A, i_B)]{i_A + i_B} C_\bullet(A+$$

⤳ LES in homology.

Cohomology:

↝ Bott-Tu

de Rham cohomology: (for smooth mnflds, using calculus)

Idea: $H^*_{singular}$ is "calculus on arbitrary top spcs"... with coefficients in <u>any</u> abelian group (not just $\mathbb{R}$ or $\mathbb{C}$) ~ b/c $H^*_{dR} \cong H^*_{sing}$

e.g. for open $U \subset \mathbb{R}^3$

$$C^\bullet_{dR}(U) = \left(\{fxns\} \xrightarrow{grad} \{vf's\} \xrightarrow{curl} \{vf's\} \xrightarrow{div} \{fxns\}\right)$$

More generally,

$$C^\infty(M;\mathbb{R})$$
$$\shortparallel$$
$$C^\bullet_{dR}(M) = \left(\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \ldots\right)$$

↙ the dR cx of M

Recall homology:

$$\begin{tikzcd} Top \arrow[r, "C_\bullet(-)"] \arrow[dr, "H_n^{sing}"'] & Ch \arrow[d, "H_n"] \\ & Ab \end{tikzcd}$$

Modified by interpolating to get homology w/ coeff in $G \in Ab$:

$$\begin{tikzcd} Top \arrow[r, "C_\bullet(-)"] \arrow[drr, "H_n(-;G)"'] & Ch \arrow[r, "(-)\otimes G"] & Ch \arrow[d, "H_n"] \\ & & Ab \end{tikzcd}$$

observe: Another operation on ch cxes:

$$C_\bullet \in Ch \rightsquigarrow \hom(C_\bullet, G)$$

$$\begin{tikzcd} \vdots \arrow[d] & & {} \\ C_{n+1} \arrow[d, "d_{n+1}"] & & \hom_{Ab}(C_{n+1};G) \arrow[u, "\delta"] \\ C_n \arrow[d, "d_n"] \arrow[rr, phantom, "\rightsquigarrow"] & & \hom_{Ab}(C_n, G) \arrow[u, "\delta"] \\ C_{n-1} \arrow[d] & & \hom_{Ab}(C_{n-1}, G) \arrow[u, "\delta"] \\ \vdots & & {} \arrow[u, "\delta"] \end{tikzcd}$$

precomposition!

Notation: $C^\vee_\bullet := \hom(C_\bullet, \mathbb{Z})$

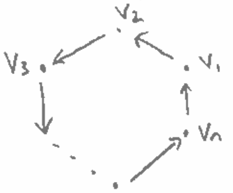
vs duality

<u>Def</u>: $H^n(C_\bullet;G) := H_{-n}(\hom(C_\bullet, G))$

$$H^n_{\star}(X;G) := H^n(C^{\star}_\bullet(X);G)$$

$\star = \{sing, cell, simp\}$.

Ex: Simplicial Cohomology of X

X = 

$$C^\Delta_\bullet(X) = \left(\mathbb{Z}\{e_1,\ldots,e_n\} \xrightarrow{d_1} \mathbb{Z}\{v_1,\ldots,v_n\}\right)$$

$e_i \longmapsto v_{i+1} - v_i \quad (i \in \mathbb{Z}/n)$

$C^\Delta_\bullet(X)^\vee$
$\shortparallel$
$C^\bullet_\Delta(X;\mathbb{Z})$
$\shortparallel$

$$\begin{tikzcd} 0 & C^\bullet_\Delta(X) = \left(\hom_{Ab}(\mathbb{Z}\{e_1,\ldots,e_n\},\mathbb{Z}) \arrow[l, "\delta_2"'] \arrow[d, phantom, "\cong"] \right. & \hom_{Ab}(\mathbb{Z}\{v_1,\ldots,v_n\},\mathbb{Z}) \arrow[l, "\delta_0"'] \arrow[d, phantom, "\cong"] & 0 \arrow[l, "\delta_1"'] \\ & \hom_{set}(\{e_1,\ldots,e_n\},\mathbb{Z}) & \hom_{set}(\{v_1,\ldots,v_n\},\mathbb{Z}) \end{tikzcd}$$

↖ w/ pointwise addition

$$e_i \mapsto \varphi(d_1(e_i)) \longleftarrow\!\shortmid \varphi$$
$$= \varphi(v_{i+1} - v_i)$$
$$= \varphi(v_{i+1}) - \varphi(v_i)$$

$$H^0_\Delta(y) = H_0(C^\bullet_\Delta(X)) := \frac{\ker(\delta_0)}{\mathrm{im}(\delta_1)} = \ker(\delta_0)$$

$$= \{\{v_1,\ldots,v_n\} \xrightarrow{\varphi} \mathbb{Z} : \delta_0(\varphi) = 0\}$$

i.e. $\forall i$ $\delta_0 \circ \varphi(e_i) = 0$

i.e. $\forall$ i, $\varphi(v_{i+1}) = \varphi(v_i)$

$\cong \mathbb{Z}$ b/c φ is constant

More generally, $H^0(X;G) \cong \hom_{set}\left(\left\{\begin{matrix}\text{set of path}\\ \text{components}\end{matrix}\right\}, G\right)$

$$H^1_\Delta(x) := H_{-1}(C^\bullet_\Delta(x)) = \frac{\ker(\delta_{-1})}{\mathrm{im}(\delta_0)} = \frac{C^1_\Delta(x)}{\mathrm{im}(\delta_0)}$$

$$= \frac{\{fxns\ \{e_1,\ldots,e_n\} \xrightarrow{\psi} \mathbb{Z}\}}{\left\{\begin{matrix}\text{can freely modify by +/- functions of the form}\\ e_i \mapsto \varphi(v_{i+1}) - \varphi(v_i) \text{ for any } \{v_1,\ldots,v_n\} \xrightarrow{\varphi} \mathbb{Z}\end{matrix}\right.}$$

observe that we can change so that value at all e_i for $i \neq 1$ is zero, then evaluate at e_1

$$= \sum_{i \in \mathbb{Z}/n} \psi(e_i) \longleftarrow\!\shortmid \psi$$

contour integral

$$\cong \mathbb{Z}$$

i.e. H^1 measures the failure of $\{e_1,\dots,e_n\} \xrightarrow{\gamma} \mathbb{Z}$ to be the "local difference in altitude" function of a function $\{v_1,\dots,v_n\} \xrightarrow{\varphi} \mathbb{Z}$

More generally, H^1(graph) measures its failure to be a tree ($\Leftrightarrow \exists \leq 1$ non-redundant edge path between any pairs of vertices)

↳ equivalence class of paths up to hmtpy rel end points.

For a differential 1-form $\omega \in \Omega^1(S^1)$

(e.g. $S^1 = I/\sim$, $\omega = dt$)

$\int_{S^1} \omega$ measures the impossibility of solving $\omega = df$ for $f \in \Omega^0(S^1) = C^\infty(S^1)$

Ex: X a 2d simplicial complex:

$$\underset{\sim}{C^0_\Delta(X)} \xrightarrow{\delta_0} C^1_\Delta(X) \xrightarrow{\delta_1} C^2_\Delta(X)$$

$$\overset{\Psi}{\Psi}$$

Vf $\vec{V}$ s.t. curl($\vec{v}$)=0

$\delta\Psi = 0$ iff Ψ is "locally additive"

If so, get $[\Psi] \in H^1_\Delta(X)$, and $[\Psi] = [0]$ iff $\exists\ \varphi \in C^0_\Delta(X)$ s.t. $\Psi = \delta\varphi$ o∞ ∃ fxn f s.t. $\vec{V} = \text{grad}(f)$

$\Updownarrow$

$$\oint_\gamma \vec{V} \cdot \gamma'(t)\,dt = 0 \quad \forall \text{ closed loops } \gamma$$

"Ψ is locally additive for trivial reasons, namely it's globally additive"

Cohomology continued:

Recall: for ch cx $C_\cdot = (\cdots \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \cdots)$, $G \in Ab$, we get a ch cx

$$\cdots \xleftarrow{\delta} \hom(C_n, G) \xleftarrow{\delta} \hom(C_{n-1}, G) \xleftarrow{\delta} \cdots$$

and homology here ⤴ is $=: H^n(C_\cdot, G)$ for the n'th cohomology group of $C_\cdot$ w/ coefficients in G.

For a space X, $H^n(X;G) := H^n(C_\cdot(X), G)$

$$\begin{tikzcd}
Top \arrow[r] \arrow[dr, dashed, "H^n(-{,}G)"'] & Ch \arrow[d] \\
 & Ab^{op}
\end{tikzcd}$$

obj: Abelian groups.
For ab gp A, B,
$\hom_{Ab^{op}}(A,B) := \hom_{AB}(B,A)$.
("the opposite" of AB)

i.e. $C_\cdot \to D_\cdot$

$$\rightsquigarrow$$

$$H^n(C_\cdot;G) \leftarrow H^n(D_\cdot;G)$$

Today: Study cohomology of ch cxes. (later cohomology of spaces)

Fact: $\{H^n(C_\cdot, G)\}_{n\in\mathbb{Z}}$ are $\underbrace{\text{determined}}$ by $\{H_n(C_\cdot)\}_{n\in\mathbb{Z}}$

"universal coefficients thm"
($\mathbb{Z}$ is the universal coeffs)

$$\Vert$$

$$H_n(C_\cdot, \mathbb{Z}) := \frac{\ker(\partial_n)}{\text{im}(\partial_{n+1})}$$

but in a subtle way.

Ex: $C_\cdot = (0 \to \underline{\mathbb{Z}} \xrightarrow{0} \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \underset{\approx}{\mathbb{Z}} \to 0)$

$H_*(C_\cdot) = \quad 0 \quad \mathbb{Z} \quad 0 \quad \mathbb{Z}/n \quad \mathbb{Z} \quad 0$

$\hom(C_\cdot, \mathbb{Z}) := C^\vee_\cdot := (0 \leftarrow \mathbb{Z} \leftarrow \mathbb{Z} \leftarrow \mathbb{Z} \leftarrow \mathbb{Z} \leftarrow 0)$

$H^{-*}(C_\cdot, \mathbb{Z}) =: H_*(C^\vee_\cdot) = \quad 0 \quad \mathbb{Z} \quad \mathbb{Z}/n \quad 0 \quad \mathbb{Z} \quad 0$

So, it's not quite true that $H^n(C_\cdot) \cong H_n(C_\cdot)^\vee$ as one might guess. However, there always exists a homomorphism

$$\begin{tikzcd}
H^n(C_\cdot; G) \arrow[r, "h"] & \hom_{Ab}(H_n(C_\cdot), G) \\
{[\varphi]} \arrow[r, mapsto] & \underline{\varphi} := h([\varphi])
\end{tikzcd}$$

$\varphi \in \ker(\partial_n) \subseteq \hom(C_n, G)$

b/c

$$\begin{tikzcd}
 & C_{n+1} \arrow[dl] \arrow[dd, "\partial_{n+1}"] \arrow[ddr, "0"] & \\
\text{im}(\partial_{n+1}) =: B_n \arrow[d, hook] \arrow[dr, hook] \arrow[drr, dotted, "0"] & & \\
\ker(\partial_n) =: Z_n \arrow[r, hook] \arrow[d] & C_n \arrow[r, "\varphi"] \arrow[d, "\partial_n"] & G \\
H_n = \frac{Z_n}{B_n} \arrow[urr, dashed, "\underline{\varphi}", "\exists!\ \text{extension}"'] \arrow[d, equal] & C_{n-1} & \\
H_n(C_\cdot) & &
\end{tikzcd}$$

$$\begin{tikzcd}
\hom(C_{n+1}, G) \\
\hom(C_n, G) \arrow[u, "\delta_n"] \\
\hom(C_{n-1}, G) \arrow[u, "\delta_{n-1}"]
\end{tikzcd}$$

Claim: the homomorphism $H^n(C_\cdot, G) \xrightarrow{h} \mathrm{hom}_{Ab}(H_n(C_\cdot), G)$ is surjective.

Pf: Observe the SES

$$\begin{tikzcd} 0 \arrow[r] & Z_n \arrow[r, hook] & C_n \arrow[r, two heads] \arrow[l, dashed, bend left] & B_{n-1} \arrow[r, two heads] \arrow[l, dashed, bend left] & 0 \end{tikzcd}$$

∃ retraction ⟺ ∃ section

("there exists a splitting of the SES: an iso w/ one of the form $0 \to A \xrightarrow{(1,0)} A \oplus B \xrightarrow{(0,1)} B \to 0$)

So, get retractions

$$\begin{tikzcd} B_n \arrow[r, hook] \arrow[dr] & Z_n \arrow[r, two heads] \arrow[d] & \frac{Z_n}{B_n} =: H_n \arrow[r, "\forall \alpha"] \arrow[d] & G \\ & C_n \arrow[r, two heads] \arrow[u, dashed, bend right] & \frac{C_n}{B_n} \arrow[u, dashed, bend right] \arrow[ur, dashed, "\exists \tilde{\alpha} \text{ extending } \alpha"'] & \end{tikzcd}$$

i.e.

$$\mathrm{Ker}(\partial_n) " \left\{ \begin{tikzcd} C_{n+1} \arrow[d, "\partial"'] \arrow[dr] \arrow[drr, bend left, "0"] & & \\ & B_n \arrow[dl] & \\ C_n \arrow[rr, dashed] & & G \end{tikzcd} \right\} = \left\{ \begin{tikzcd} B_n \arrow[d] \arrow[dr, "0"] & \\ C_n \arrow[r, dashed] & G \end{tikzcd} \right\} = \left\{ \begin{tikzcd} \frac{C_n}{B_n} \arrow[r, dashed] & G \end{tikzcd} \right\}$$

$$\begin{tikzcd} \mathrm{Ker}(\partial_n) \arrow[d, two heads] \arrow[dr, "\text{This is surjective!}"] & \\ \frac{\mathrm{Ker}(\delta_n)}{\mathrm{im}(\delta_{n-1})} =: H^n(C_\cdot, G) \arrow[r, "h"] & \mathrm{hom}_{Ab}(H_n, G) \end{tikzcd}$$

So h is surjective too. ∎

Better, for a fixed retraction $\begin{tikzcd} \frac{Z_n}{B_n} =: H_n \arrow[d, hook] \\ \frac{C_n}{B_n} \arrow[u, dashed, bend right, "r"'] \end{tikzcd}$, we get

$$\begin{tikzcd} \underbrace{H^n(C_\cdot;G)}_{\text{object of interest ...}} \arrow[r, two heads, "h"'] & \underbrace{\mathrm{hom}(H_n, G)}_{\text{described partly in terms of this.}} \arrow[l, dashed, bend right, "{[r \circ (-)]} \leftsquigarrow \text{section, a hom.}"'] \end{tikzcd}$$

Have a splittable SES

$$\begin{tikzcd} 0 \arrow[r] & \underbrace{\ker(h)}_{} \arrow[r, hook] & H^n(C_\cdot, G) \arrow[r, two heads, "h"'] & \mathrm{hom}(H_n(C_\cdot), G) \arrow[r] \arrow[l, dashed, bend right] & 0 \end{tikzcd}$$

Q: What is this? (Towards fully understanding

$$H^n(C_\cdot, G) \overset{\exists}{\cong} \ker(h) \oplus \mathrm{hom}(H_n(C_\cdot), G)$$

by splittable SES.

Cohomology continued:

$C_\cdot \in Ch$, $G \in Ab \rightsquigarrow H^n(C_\cdot;G) := H_n(\mathrm{hom}(C_\cdot, G))$

and $H^n(X;G) := H^n(C_\cdot(X);G)$ (simp, sing, cell)

Towards Universal Constant Theorem:

$$\begin{tikzcd} Top \arrow[r, "C_\cdot(-)"] & Ch \arrow[r, "H^*(-;G)"] \arrow[d, "H_*(-)"'] & gAb^{op} \\ H_*(-;\mathbb{Z}) \arrow[ur, phantom, "="] & gAb \arrow[ur, dotted] & \end{tikzcd}$$

↑ only depends on homology with $\mathbb{Z}$ coeffs!

observe the SES: (of Ch cxes)

$Z_n = Z_n(C_\cdot)$

$B_n = B_n(C_\cdot)$

$H_n = H_n(C_\cdot)$

$$\begin{tikzcd} & \vdots \arrow[d, "0"'] & \vdots \arrow[d, "\partial"] & \vdots \arrow[d, "0"] & \\ 0 \arrow[r] & Z_{n+1} \arrow[r, hook] \arrow[d, "0"'] & C_{n+1} \arrow[r, two heads, "\partial"] \arrow[d, "\partial"] & B_n \arrow[r] \arrow[d, "0"] & 0 \\ 0 \arrow[r] & Z_n \arrow[r, hook] \arrow[d, "0"'] & C_n \arrow[r, two heads, "\partial"] \arrow[d, "\partial"] & B_{n-1} \arrow[r] \arrow[d, "0"] & 0 \\ & \vdots & {} & {} & \end{tikzcd}$$

$\Big\downarrow$ $\mathrm{hom}(-, G)$

$$\begin{tikzcd} & \vdots & \vdots & \vdots & \\ 0 & \mathrm{hom}(Z_{n+1}, G) \arrow[l] \arrow[u, "0"] & \mathrm{hom}(C_{n+1}, G) \arrow[l, two heads] \arrow[u] & \mathrm{hom}(B_n, G) \arrow[l, hook] \arrow[u, "0"'] & 0 \arrow[l] \\ 0 & \mathrm{hom}(Z_n, G) \arrow[l] \arrow[u, "0"] & \mathrm{hom}(C_n, G) \arrow[l, two heads] \arrow[u] & \mathrm{hom}(B_{n-1}, G) \arrow[l, hook] \arrow[u, "0"'] & {} \arrow[l] \\ & \vdots \arrow[u, "0"] & \vdots \arrow[u] & \vdots \arrow[u, "0"'] & \end{tikzcd}$$

$H_{-n}(\text{this}) =: H^n(C_\cdot;G)$

differentials all zero so n'th homology = n^th group

$\Big\downarrow$ H_* LES

$$\begin{tikzcd} \cdots & \mathrm{hom}(B_n, G) \arrow[l] & \\ \mathrm{hom}(Z_n, G) \arrow[ur, "i_n^*"] & H^n(C_\cdot;G) \arrow[l] & \mathrm{hom}(B_{n-1}, G) \arrow[l] \\ & & \cdots \arrow[u, "i_{n-1}^*"'] \end{tikzcd}$$

⊛ Aside: given SES $0 \to X \hookrightarrow Y \twoheadrightarrow Z \to 0$

only get ES:

$$\mathrm{hom}(X,G) \leftarrow \mathrm{hom}(Y,G) \hookleftarrow \mathrm{hom}(Z,G) \leftarrow 0$$

↑ lose surjectivity.

$$\begin{tikzcd} B_n \arrow[r, hook, "i_n"] & Z_n \arrow[d, dashed] \\ & G \end{tikzcd}$$

(precomp w/ i_n)

$$\Big\downarrow \text{SES}$$

(**)
$$0 \to \underbrace{\operatorname{coker}(i_{n-1}^*)} \hookrightarrow H^n(C_\cdot;G) \overset{h}{\twoheadrightarrow} \underbrace{\ker(i_n^*)} \to 0$$

$$\hom_{Ab}(H_n(C_\cdot),G) =: \left\{ H_n := \frac{Z_n}{B_n} \dashrightarrow G \right\} \cong \left\{ \begin{tikzcd} B_n \arrow[d, hook, "i_n"'] \arrow[dr, "0"] & \\ Z_n \arrow[r, dashed] & G \end{tikzcd} \right\} =:$$

$$\frac{\{B_{n-1} \dashrightarrow G\}}{\left\{ \begin{tikzcd} B_{n-1} \arrow[r] \arrow[d] & G \\ Z_{n-1} \arrow[ur, dashed, "\exists \text{ext}"'] & \end{tikzcd} \right\}}$$

≈ "the obstruction group to the existence of extensions!"

observe the SES $0 \to B_{n-1} \hookrightarrow Z_{n-1} \twoheadrightarrow H_{n-1}(C_\cdot) \to 0$

Again, note only get ES:
⊛

$$\hom(B_{n-1},G) \xleftarrow{i_{n-1}^*} \hom(Z_{n-1},G) \hookleftarrow \hom(H_{n-1},G) \leftarrow 0$$

↖ not generally surjective

Idea: Consider this SES as a quasi iso:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & B_{n-1} \arrow[r, hook] \arrow[d] & Z_{n-1} \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & 0 \arrow[r] & H_{n-1} \arrow[r] & 0 \end{tikzcd}$$

Both free Ab gps.

⇝ This is a free resolution of H_{n-1}

(In general, a free resolution is a gp from a complex of free ab gps.)

Def/Prop: For any $A, G \in Ab$ and any free resolution $F_\cdot \xrightarrow{\simeq} \underset{\sim}{A}$, the Ext(ension) groups, aka the right derived functors of hom, are

$$\operatorname{Ext}^n_{\underset{Ab}{\cap}}(A,G) := \mathbb{R}^n\hom(A,G) := H_{-n}(\hom(F_\cdot,G)). \quad (n \geq 0)$$

---

i.e.

$$\begin{tikzcd} \cdots \arrow[r] & F_2 \arrow[r] \arrow[d] & F_1 \arrow[r] \arrow[d] & \underset{\sim}{F_0} \arrow[r] \arrow[d] & 0 \\ \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \underset{\sim}{A} \arrow[r] & 0 \end{tikzcd} \rightsquigarrow \cdots \xleftarrow{\delta^2} \hom(F_2,G) \xleftarrow{\delta^1} \hom(F_1,G) \xleftarrow{\delta^0} \hom(F_0,G) \leftarrow 0$$

Using ⊛, ES $\cdots \to F_1 \to F_0 \to A \to 0$

⇝ ES $\hom(F_1,G) \xleftarrow{\delta^0} \hom(F_0,G) \hookleftarrow \hom(A,G) \leftarrow 0$

So $\operatorname{Ext}^0 := \ker(\delta^0) = \hom(A,G)$
$\shortparallel$
hom (A,G)

Returning to our SES (**), we now have a name for the Kernel: it's

$$\operatorname{coker}(i_{n-1}^*) = H_{-1}(\hom(\underset{\substack{\shortparallel \\ (B_{n-1} \hookrightarrow \underset{\sim}{Z_{n-1}})}}{F_\cdot}, G)) =: \operatorname{Ext}^1(H_{n-1}(C_\cdot), G)$$

In particular, by Prop. this only depends on $H_{n-1}(C_\cdot)$.

I.e. we have (algebraic) universal coeff thm: For $C_\cdot \in Ch$ a ch cx of levelwise free ab grps, have a noncanonical split SES

$$0 \to \operatorname{Ext}^1(H_{n-1}(C_\cdot),G) \hookrightarrow H^n(C_\cdot;G) \twoheadrightarrow \hom(H_n(C_\cdot,G)) \to$$

(We'll apply this to $C_\cdot^{sing}(X)$, $C_\cdot^{\Delta}(X)$, or $C_\cdot^{cell}(X)$, all of which are l.w.-free.)

Aside: Any ab gp has a two-term resolution:

$$\begin{tikzcd} 0 \arrow[r] & F_1 \arrow[r, hook] \arrow[d] & \underset{\sim}{F_0} \arrow[d] \\ 0 \arrow[r] & 0 \arrow[r] & \underset{\sim}{A} \end{tikzcd}$$

So by Prop:

$$\operatorname{Ext}^{\geq 2}(A,G) = 0 \quad \forall A, G \in Ab!$$

However, have the same notion for R-modules for any ring R Ext^n_R, (i.e. $\operatorname{Ext}^n_R := \operatorname{Ext}^n_{\mathbb{Z}}$), and Ext^n_R may be nonzero for $n \geq 2$.

"The ring $\mathbb{Z}$ has homological dimension 1"

If $R = F$ is a field, then all F-modules are free, hence they are their own free resolution, and $\operatorname{Ext}^n_F = 0 \ \forall \ n \geq 0$

"A field F has homological dimension 0"

Prop: ① $\operatorname{Ext}(A \oplus A', G) \cong \operatorname{Ext}(A,G) \oplus \operatorname{Ext}(A',G)$

(Pf: for free resolutions $F_\cdot \xrightarrow{\simeq} A$ and $F_\cdot' \xrightarrow{\simeq} \underset{\sim}{A'}$, get a free resolution $F_\cdot \oplus F_\cdot' \xrightarrow{\simeq} A \oplus A'$, hence

$$\hom(F_\cdot \oplus F_\cdot', G) \cong \hom(F_\cdot, G) \oplus \hom(F_\cdot', G$$

Then take H_{-1}.)

② If $F \in Ab$ is <u>free</u> then $\mathrm{Ext}(F,G) := \mathrm{Ext}^1_{\mathbb{Z}}(F,G) = 0$

(Pf: $\underline{F}$ is a free resolution of itself)

③ $\mathrm{Ext}(\mathbb{Z}/n, G) = G/nG$.

(Pf: Take free resolution
$$\begin{tikzcd} F_\bullet \arrow[d, phantom] & (\mathbb{Z} \arrow[r, hook, "n"] \arrow[d] & \underset{\sim}{\mathbb{Z}}) \arrow[d, two heads] & 1 \arrow[d] \\ & 0 \arrow[r] & \mathbb{Z}/n & 2 \end{tikzcd}$$

Then:

$$\hom(F_\bullet, G) = \left(\hom(\mathbb{Z},G) \xleftarrow{n^*} \underset{\sim}{\hom(\mathbb{Z},G)} \right)$$

$$= \left(G \xleftarrow{n} \underset{\sim}{G} \right)$$
$$ng \longleftarrow\!\shortmid g$$

so $\mathrm{Ext}(\mathbb{Z}/n, G) := H_{-1}(\;) = G/nG$

Corr: If A is finitely generated, then $\mathrm{Ext}(A,\mathbb{Z}) = \mathrm{Tors}(A)$, the torsion subgroup:

$\{a \in A : \exists n \geq 1 \text{ s.t. } n \cdot a = 0\}$

Corr: $(G=\mathbb{Z})$: For $C_\bullet$ levelwise free, whe have a noncanonically splittable SES:

$$0 \to \mathrm{Tors}(H_{n-1}(C_\bullet)) \hookrightarrow H^n(C_\bullet;G) \twoheadrightarrow \underbrace{\frac{H_n(C_\bullet)}{\mathrm{Tors}(H_n(C_\bullet))}}_{\substack{\hom(A,\mathbb{Z}) \\ \cong A/\mathrm{Tors}(A)}} \to 0$$

i.e. $H^n(C_\bullet;\mathbb{Z}) \cong \mathrm{Free}(H_n(C_\bullet)) \oplus \mathrm{Tors}(H_{n-1}(C_\bullet))$.

Ex: Recall $\mathbb{R}P^n = e^0 \cup e^1 \cup e^2 \cdots \cup e^n$

$$C^{cell}_\bullet(\mathbb{R}P^n) = \left(\cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \underset{\sim}{\mathbb{Z}} \right)$$

$$H_i(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z} & i=n \text{ if } n \text{ odd} \\ \mathbb{Z}/2 & 0<i<n \quad \text{odd} \\ 0 & \text{o.w.} \end{cases} \qquad H^i(\mathbb{R}P^n;\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & 0 \leq i \leq n \\ 0 & \text{o.w.} \end{cases}$$

because $\mathrm{Ext}(\mathbb{Z},\mathbb{Z}/2) \overset{②}{=} 0$ and $\mathrm{Ext}(\mathbb{Z}/2,\mathbb{Z}/2) \overset{③}{=} \mathbb{Z}/2$

☆ As a ring, $H^*(\mathbb{R}P^n;\mathbb{Z}/2) = (\mathbb{Z}/2)[x]/x^{n+1}$, $\deg(x) = 1$. ¯_ツ_/¯

Corr: If $C_\bullet \xrightarrow{\simeq} D_\bullet$ a q.i. b/w levelwise free ch cxes, then

$$H^n(C_\bullet;G) \xleftarrow{\cong} H^n(D_\bullet;G) \quad \forall n \in \mathbb{Z}, \forall G \in Ab.$$

(i.e. $\hom(C_\bullet,G) \xleftarrow{\simeq} \hom(D_\bullet,G)$).

Cohomology of Spaces

<u>Defs</u>: $X \in Top$ $\quad C^\bullet_{sing}(X;G) := \underbrace{\hom(\overbrace{C^{sing}_\bullet(X)}^{\text{in deg} \geq 0}, G)}_{\text{in deg} \leq 0}$

$$H^n_{sing}(X;G) := H_{-n}(C^\bullet_{sing}(X;G)).$$

Similarly, for H^n_Δ, H^n_{cell}.

Observe: $C^\Delta_\bullet(X) \xhookrightarrow{\simeq} C^{sing}_\bullet(X)$ and both are lw free.

So, by corr $H^n_\Delta(X;G) \xleftarrow{\cong} H^n_{sing}(X;G)$

Less trivially, we <u>also</u> have

$$H^n_{cell}(X;G) \xleftarrow{\cong} H^n_{sing}(X;G)$$
$$\parallel$$
$$H_{-n}\Big(\underbrace{\hom(\overbrace{C^{cell}_\bullet(X)}^{\substack{\text{dim } n: \\ \mathbb{Z}^{\oplus\left(\substack{n\text{-cells} \\ \text{of } X}\right)}}}, G)}_{}\Big)$$

$$\hom_{Ab}\left(\overset{\text{dim } -n:}{\mathbb{Z}^{\oplus\left(\substack{n\text{-cells} \\ \text{of } X}\right)}}, G\right) \cong \hom_{set}\left(\left\{\substack{n\text{-cells} \\ \text{of } X}\right\}, G\right)$$
$$\wr\wr$$
$$C^{n+1}_{cell}(X;G) \xleftarrow{\delta} C^n_{cell}(X;G)$$
$$\parallel$$

"transposed $\underbrace{\text{matrix of integers}}_{\substack{\text{degrees of attaching maps} \\ \text{for } (n+1) \text{ cells.}}}$, but now acting on G"

Terminology:

$$\begin{tikzcd} C_\bullet(X): \cdots \arrow[r, "\partial_{n+2}"] & C_{n+1}(X) \arrow[r, "\partial_{n+1}"] & C_n(X) \arrow[r, "\partial_n"] & C_{n-1}(X) \arrow[r, "\partial_{n-1}"] & \cdots \end{tikzcd}$$

$\downarrow \{\hom(-,G)$ $\qquad$ $H^n(X;G) :=$ the homology here

$$\begin{tikzcd} C^\bullet(X;G): \cdots & C^{n+1}(X;G) \arrow[l, "\delta^{n+1}"'] & C^n(X;G) \arrow[l, "\delta^n"'] & C^{n-1}(X;G) \arrow[l, "\delta^{n-1}"'] & \cdots \arrow[l] \end{tikzcd}$$

coboundary map

Cocycles:

$$Z^n(X;G) := \mathrm{Ker}(\delta^n) := \left\{ \begin{tikzcd} C_{n+1}(X) \arrow[d, "\partial"'] \arrow[dr, "0"] & \\ C_n(X) \arrow[r, dashed] & G \end{tikzcd} \right\}$$

$\cong$ {functions $\hom(\Delta^n, X) \to G$ that vanish on the boundaries of singular $(n+1)$ simp's in X}

Coboundaries:

$$B^n(X;G) := \mathrm{im}(\delta^{n-1}) := \left\{ \begin{tikzcd} C_n \arrow[r, dashed] \arrow[d, "\partial"'] & G \\ C_{n-1}(X) \arrow[ur] & \end{tikzcd} \right\}$$

$\cong$ {functions $\hom(\Delta^n, X) \to G$ that are determined by their values on boundaries.}

Note: $C_\cdot^{sing}$, $C_\cdot^{cell}$, $C_\cdot^{\Delta}$ are levelwise free, so we can apply the (algebraic) universal constant thm:

$$0 \to \mathrm{Ext}(H_{n-1}(X),G) \hookrightarrow H^n(X;G) \twoheadrightarrow \hom(H_n(X),G) \to 0$$

☆ At $n=0$, $H_{-1}(X)=0$, so $H^0(X,G) \xrightarrow{\cong} \underbrace{\hom(\underbrace{H_0(X)}_{\mathbb{Z}^{\oplus\{\text{path comp.}\}}}, G)}_{\mathrm{Ker}(\delta^0) \cong \hom_{set}(\{\text{path comp}\}, G)}$

☆ At $n=1$, $H_{n-1}(X)$ is free, so

$$H^1(X;G) \xrightarrow{\cong} \hom(H_1(X),G).$$

special case of $G=F$ a field, the same considerations show: $\mathrm{Ext}_F^{>0}=0$ (since all F-modules are free)

$$H^n(X;F) \xrightarrow{\cong} \hom_F(H_n(X;F),F) =: H_n(X;F)^\vee$$

$$H_n(X;F) = H_n(C_\cdot(X) \underset{\mathbb{Z}}{\otimes} F)$$

Consistency check: Computed $H_i(\mathbb{RP}^n;\mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & 0 \le i \le n \\ 0 & \text{o.w.} \end{cases}$

Reduced Cohomology:

$$\tilde{C}^\cdot(X;G) := \hom(\underbrace{\tilde{C}_\cdot(X)}, G)$$

"sum of coeffs"

$$\cdots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} \underline{C_0(X)} \xrightarrow{\varepsilon} \mathbb{Z}$$

observe:

$$\begin{tikzcd} \tilde{C}_\cdot(X) \arrow[r] & C_\cdot(X) \arrow[d, "\text{dualize}"] \\ \tilde{C}^\cdot(X,G) & \tilde{C}^\cdot(X,G) \arrow[l] \end{tikzcd}$$

Clearly, $\tilde{H}^n(X;G) \xleftarrow{\cong} H^n(X;G)$ for $n \ge 1$.

Recall:
$$\begin{tikzcd} \tilde{H}_0(X) \arrow[r, hook] \arrow[d, equal] & H_0(X) \arrow[d, equal] \\ \{\sum_i n_i[p_i] : \sum n_i = 0\} & \mathbb{Z}^{\oplus\{\text{path cpts}\}} \end{tikzcd}$$

Dually:
$$\begin{tikzcd} \tilde{H}^0(X;G) & H^0(X;G) \arrow[l, two heads] \arrow[d, equal] \\ & \{\text{fxns } \{\text{path cpts}\} \to G\} \end{tikzcd}$$

$\mathrm{Ker} = \{\underline{\text{constant}} \text{ fns } \{\text{path cpts}\} \to G\}$

Relative LES:

Have SES:

$$\begin{tikzcd} 0 \arrow[r] & C_n(A) \arrow[r, hook] & C_n(X) \arrow[r] \arrow[l, dashed, bend right, "\exists \text{ retraction}"'] & C_n(X,A) \arrow[r] & 0 \end{tikzcd}$$

True because these are free, and we include generators. A map of degree two for example does not admit a retraction b/c $n \mapsto 1$, what do the rest do? no free.

⤳ get SES:

$$0 \leftarrow C^n(A,G) \leftarrow C^n(X;G) \leftarrow C^n(X,A;G) \leftarrow 0$$

↑ surj! (because of retraction above), i.e.

$$\begin{tikzcd} C_n(A) \arrow[d, "c"] & C_n(X) \arrow[l, "r"'] \\ G & \end{tikzcd}$$

pre comp w/ r

so, SES:

$$0 \leftarrow C^\cdot(A;G) \leftarrow C^\cdot(X;G) \leftarrow C^\cdot(X,A;G) \leftarrow 0$$

⤳ LES in hlgy:

$$\cdots \leftarrow H^{n+1}(X;G) \leftarrow H^{n+1}(X,A;G) \xleftarrow{\delta} H^n(A;G)$$

$\cong H^{n+1}(X/A;G)$ for $\underline{\text{good}}$ pairs

In fact, this is closely related to $H_{n+1}(X,A) \xrightarrow{\partial} H_n(A)$

$$\begin{tikzcd} H^n(A;G) \arrow[r, "\delta"] \arrow[d, "h"'] & H^{n+1}(X,A;G) \arrow[d, "h"] \\ \hom(H_n(A);G) \arrow[r, "\partial^*"'] & \hom(H_{n+1}(X,A),G) \end{tikzcd}$$

(i.e. precomp w/ ∂)

Induced homs:

Given $C_\cdot \to D_\cdot$ in Ch

(e.g. $C_\cdot(X) \xrightarrow{C_\cdot(f)} C_\cdot(Y)$) for $X \xrightarrow{f} Y$ in Top.

get

$$\hom(C_\cdot,G) \leftarrow \hom(D_\cdot,G)$$

$$\rightsquigarrow H^n(C_\cdot;G) \leftarrow H^n(D_\cdot;G)$$

i.e. cohomology is a $\underline{\text{contravariant}}$ functor:

$$\begin{tikzcd} Ch \arrow[r, "H^n(-;G)"] & Ab^{op} \\ Top \arrow[u, "C_\cdot(-)"] \arrow[ur, "H^n_{sg}(-;G)"'] & \end{tikzcd}$$

$C_\cdot \in Ch \rightsquigarrow H^n(C_\cdot;G) \to \hom(H_n(C_\cdot),G)$

a $\underline{\text{natural}}$ transformation

$$\begin{tikzcd} Ch^{op} \arrow[rr, bend left, "H^n(-;G)"] \arrow[dr, "H_n(-)"'] & \Downarrow & Ab \\ & Ab^{op} \arrow[ur, "\hom(-;G)"'] & \end{tikzcd}$$

Homotopy invariance:

Fact: $\hom(\overset{ch}{htpy};G) = \overset{ch}{htpy}$. So, ch-htpc maps induce the same map in coh of ch cxes, hence homotope maps b/w Top spaces induce the same map in Ch.

Excision: For $Z \subset A \subset X$, $\bar{Z} \subseteq \mathring{A}$,

$$(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$$

induces iso's on relative homology, hence on coh $\forall$ coeffs G (by UCT).

Mayer Vietoris:

$A, B \subseteq X$, $\mathring{A} \cup \mathring{B} = X$

Recall SES:

$$\begin{tikzcd} & & & C_\cdot(X) \\ 0 \arrow[r] & C_\cdot(A \cap B) \arrow[r, hook] & C_\cdot(A) \oplus C_\cdot(B) \arrow[r, two heads] & C_\cdot(A+B) \arrow[r] \arrow[u, "\simeq"] & 0 \end{tikzcd}$$

All are lw-free, so get SES in ch on $\hom(-, G)$,

hence LES:

$$\curvearrowright \cdots H^n(A \cap B; G) \leftarrow H^n(A, G) \oplus H^n(B; G) \leftarrow H^n(X; G))$$

§ 3.2 Cup Product:

Now, study cohomology with coeffs in a ring R (usually will be commutative, usually $\mathbb{Z}, \mathbb{Z}/n, \mathbb{Q}, \mathbb{C} \ldots$)

Note all of

$$\begin{tikzcd} \hom_{Ab}(C_k(X), R) =: C^k(X; R) & Z^k(X; R) \arrow[l, hook'] \arrow[d] & B^k(X; R) \arrow[l, hook'] \\ & H^k(X; R) & \end{tikzcd}$$

takes place in R-modules.

Def: Given $\varphi \in C^k(X; R)$ and $\psi \in C^\ell(X; R)$, their cup product is $\varphi \cup \psi \in C^{k+\ell}(X; R)$ is defined by:

$$(\varphi \cup \psi)(\underbrace{\sigma) := \varphi(\sigma|_{\Delta^{\{0, \ldots, k\}}}}_{\text{a generator of } C_{k+\ell}(X) = \mathbb{Z}\{\hom_{Top}(\Delta^{k+\ell}, X)\}}) \cdot \psi(\sigma|_{\Delta^{\{k+1, \ldots, k+\ell\}}})$$

multiplication in R

Lemma: $\delta(\varphi \cup \psi) = (\delta\varphi) \cup \psi + (-1)^k \cdot \varphi \cup \delta(\psi)$

"graded" Leibniz rule

Observations: Cup products are:

- Associative
- Multilinear, e.g. $(\varphi_1 + \varphi_2) \cup \psi = \varphi_1 \cup \psi + \varphi_2 \cup \psi$
  $\hat{R}$

Corr: ✭ If φ & ψ are cocycles, so is $\varphi \cup \psi$

✭ If moreover either φ or ψ is a coboundary, then so is $\varphi \cup \psi$.

So, we get

$$H^k(X; R) \underset{R}{\otimes} H^\ell(X; R) \longrightarrow H^{k+\ell}(X; R)$$

for R commutative.

Altogether,

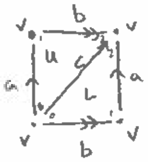
$$H^*(X; R) := \bigoplus_{i \geq 0} H^i(X; R)$$

becomes a graded R-algebra.

Thm: (3.11) This is in fact graded-commutative:

$$x \cdot y = (-1)^{\deg(x) \cdot \deg(y)} y \cdot x$$

for pure-dimensional elements x & y.

Ex: $X = T^2 =$  $R = \mathbb{Z}$

$$\left.\begin{array}{l} \partial u = b - c + a \\ \partial L = a - c + b \end{array}\right. \rightsquigarrow H_2 = \mathbb{Z}\{[u - L]\}$$

$$H_1 = \mathbb{Z}\{[a], [b]\} \ni [c] = [a] + [b]$$

UCT:

$$0 \to \underbrace{\operatorname{Ext}(H_0, \mathbb{Z})}_{\substack{\text{free} \\ 0}} \hookrightarrow H^1 \underset{\cong}{\twoheadrightarrow} \hom(H_1, \mathbb{Z}) \to 0 \qquad \Rightarrow$$

$$\hom(H_1, \mathbb{Z}) = \mathbb{Z}\{[\alpha], [\beta]\}$$

dual basis: $[\alpha]_i : \begin{array}{l} [a] \to 1 \\ [b] \to 0 \end{array}$ $\quad [\beta]_i : \begin{array}{l} [a] \to 0 \\ [b] \to 1 \end{array}$

$$0 \to \underset{\substack{\| \\ 0}}{\operatorname{Ext}(H_1, \mathbb{Z})} \hookrightarrow H^2 \underset{\cong}{\twoheadrightarrow} \hom(H_2, \mathbb{Z}) \to 0 \qquad \Rightarrow$$

✭ Only possibly nontrivial cup product is

$$H^1 \times H^1 \longrightarrow H^2$$

Must lift $[\alpha], [\beta] \in H^1$ to cocycles $\alpha, \beta \in Z^1 \twoheadrightarrow H^1 = \frac{Z^1}{B^1}$

$$Z^1 \subset C^1 := \hom_{Ab}(C_1, \mathbb{Z}) :: \hom_{Set}(\underbrace{\{a, b, c\}}_{\substack{:: \\ X_1}}, \mathbb{Z})$$

1 cocycles: $\{\varphi \in C^1 \text{ s.t. } \delta\varphi = 0\}$

$:= \{\varphi \in C^1 \text{ s.t. } \forall \sigma \in C_2,$
$\varphi(\partial\sigma) =: (\delta\varphi)(\sigma) = 0\}$

Observe: $[\alpha] \in H^1$ is represented by $X_1 \xrightarrow{\alpha} \mathbb{Z}$

Need:

$0 \overset{!}{=} \alpha(\partial u) := \alpha(b - c + a) = 0 - \alpha(c) + 1$

$0 \overset{!}{=} \alpha(\partial L) := \alpha(a - c + b) = 1 - \alpha(c) + 0$

$a \to 1$
$b \to 0$
$c \to 1$!

For β, $\begin{array}{l} a \to 0 \\ b \to 1 \\ c \to 1 \end{array}$!

Note: this can be observed from $[c] = [a] + [b]$

Now compute:

$[\alpha]\smile[\alpha], [\alpha]\smile[\beta], [\beta]\smile[\alpha], [\beta]\smile[\beta] \in H^2 \cong \hom(\mathbb{Z}\{[U-L]\}, \mathbb{Z}) \cong \mathbb{Z}$

- $([\alpha]\smile[\alpha])([U-L]) := (\alpha\smile\alpha)(U-L)$

  $\alpha\smile\alpha : U \to \alpha(U|_{\Delta^{\{0,1\}}}) \cdot \alpha(U|_{\Delta^{\{1,2\}}}) = \alpha(a)\cdot\alpha(b) = 0$

  $L \to$ —— 〃 —— $= \alpha(b)\cdot\alpha(a) = 0$

- By similar logic, $\beta\smile\beta = 0$

- $\alpha\smile\beta : U \to \alpha(U|_{\Delta^{\{0,1\}}}) \cdot \beta(U|_{\Delta^{\{1,2\}}}) = \alpha(a)\cdot\beta(b) = 1$

  $L \to$ —— 〃 —— $\alpha(b)\cdot\beta(a) = 0$

  So $([\alpha]\smile[\beta])([U-L]) = (\alpha\smile\beta)(U-L) = 1 - 0 = 1$

  i.e. $[\alpha]\smile[\beta] \in H^2 \cong \hom(H_2, \mathbb{Z})$ is a generator!

- $\beta\smile\alpha : U \to 0$

  $L \to 1$

  Check graded commutativity:

  $[\alpha]\smile[\beta] = (-1)^{1\cdot 1}[\beta]\smile[\alpha]$ in $H^2 \cong \hom(H_2, \mathbb{Z})$

Altogether: $H^*(T^2;\mathbb{Z}) \cong \mathbb{Z}[\alpha,\beta] \big/ \alpha^2 = \beta^2 = 0$ $=: \Lambda_{\mathbb{Z}}[\alpha,\beta]$, the exterior algebra.

$\rightsquigarrow$

For gr-comm ring on α, β in deg 1,

$(\Rightarrow \alpha\cdot\beta = -\beta\cdot\alpha)$

Remark: In <u>any</u> gr-comm ring A, $a \in A$ has precisely one odd degree, $a\cdot a = (-1)^{\deg(a)\cdot\deg(a)} a\cdot a = -a\cdot a \Leftrightarrow 2(a\cdot a) = 0$

(so $a\cdot a = 0$ if $2 \in A^\times$ or more generally if $2 \circlearrowright A$ is injective).

Poincaré Duality!

Def: An n-manifold is a topological space M that's Hausdorff and locally homeomorphic to $\mathbb{R}^n$. (w/ bdry $\mathbb{R}^n$ or $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$).

Recall local homology at $x \in M$:

$$H_i(M|x) := H_i(M|x;\mathbb{Z}) := H_i(M, M\setminus\{x\}, \mathbb{Z}) \overset{\text{excision}}{\cong} H_i(\underset{\text{open } \mathbb{R}^n \text{ nbhd of } x}{U}, U\setminus\{x\}, \mathbb{Z})$$

$$\xrightarrow[\cong]{\text{LES}} \tilde{H}_{i-1}(U\setminus\{x\};\mathbb{Z}) \underset{\text{htpy invariance}}{\cong} \tilde{H}_{i-1}(S^{n-1};\mathbb{Z}) \underset{\text{abstractly}}{\cong} \begin{cases} \mathbb{Z} & i=n \\ 0 & \text{o.w} \end{cases}$$

i.e. $H_n(M|x)$ is a free rk-1 ab gp.

obs. for a ball $B \subsetneq \mathbb{R}^n$, $H_i(\mathbb{R}^n|B) = \begin{cases} \mathbb{Z} & i=n \\ 0 & \text{o.w} \end{cases}$

Def: An orientation of M is an assignment

$$\begin{tikzcd} M \arrow[r, mapsto] & H_n(M|x) \\ x \arrow[u, phantom, sloped, "\in"] & \mu_x \text{ a generator} \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

That is "locally consistent," i.e. $\forall\, \mathbb{R}^n \cong U \overset{\text{open}}{\subseteq} M$, $\forall$ ball $B \subsetneq U \cong \mathbb{R}^n$, $\exists\, \mu \in H_n(U|B)$ s.t. $\forall x \in B$

$$\begin{tikzcd} H_n(U|B) \arrow[r, "\cong"] & H_n(U|x) \cong H_n(M|x) \\ \mu \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto, "\cong"] & \mu_x \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

→ can be generalized to R-orientation for R-modules.

Construction: $M_R := \{\text{pairs } (x \in M, \alpha_x \in H_n(M|x;R))\}$ topologized so that $\forall\, \mathbb{R}^n \cong U \overset{\text{open}}{\subseteq} M$, $\forall\, B \subsetneq \mathbb{R}^n$, $\forall\, \alpha_B \in H_n(U|B, R)$, the subset

$$\{(x \in U, \underset{\alpha_B}{\text{image of}} \in H_n(M|x;R)\} \text{ is open.}$$

observation: R-orientations of M are equivalent to sections →

$$\begin{tikzcd} & M_R \arrow[d] \\ M \arrow[ur, dotted] \arrow[r, "="'] & M \end{tikzcd}$$

of $M_R \to M$ that pick out a generator of $H_n(M|x;R) \in \mathrm{Mod}_R$, $\forall x \in M$.

Ex: $M_R \supseteq M_0 := \{(x, \alpha_x) : \alpha_x = 0\}$

$$\begin{tikzcd} M_R \arrow[d] & M_0 \arrow[dl, "\cong \text{ homeo.}"] \\ M \end{tikzcd}$$

More generally, for any $r \in R$

$$M_R \supseteq M_r := \{(x \in M, \underbrace{\left(\substack{\text{generator} \\ \text{of } H_n(M|x)}\right) \otimes r}_{\in H_n(M|x) \otimes R \underset{\text{canon}}{\cong} H_n(M|x;R)})\}.$$

In fact, $M_r = M_{-r}$ and moreover

$$M_R = \bigsqcup_{\underbrace{[r] \in R/(r \sim -r \; \forall r \in R)}_{\text{just a set!}}} M_r$$

☆ An R-orientation, thought of as a section

$$\begin{tikzcd} & M_R \arrow[d] \\ M \arrow[ur, dashed] \arrow[r, "="'] & M \end{tikzcd}$$

must land in $M_u \subseteq M_R$ for some $u \in R^\times$ (assuming M connected).

Ex: $M = \mathbb{RP}^2$, then

$$M_{\mathbb{Z}} = \bigsqcup_{n \geq 0} M_n = \underbrace{\mathbb{RP}^2}_{n=0} \sqcup \underbrace{\bigsqcup_{n \geq 1} S^2}_{\text{double cover } S^2 \to \mathbb{RP}^2.}$$

$$\begin{tikzcd} M_{\mathbb{Z}} \arrow[d] & = & \mathbb{RP}^2 \arrow[dll, "\cong"'] & \sqcup & \bigsqcup_{n \geq 1} S^2 \arrow[dllll, bend left, "2 \text{ to } 1"] \\ \mathbb{RP}^2 \end{tikzcd}$$

— $n=1$ copy = subset of M_u for $u \in \mathbb{Z}^\times = \{\pm 1\}$

So, $\nexists$ section of $\begin{smallmatrix} M_{\mathbb{Z}} \\ \downarrow \\ M \end{smallmatrix}$ that lands in generators in each fiber. i.e. $\mathbb{RP}^2$ is not $\mathbb{Z}$ orientable.

On the other hand

$$\begin{tikzcd} M_{\mathbb{Z}/2} = M_{[0]} \sqcup M_{[2]} \cong \mathbb{RP}^2 \sqcup \mathbb{RP}^2 \arrow[d] \\ \mathbb{RP}^2 \end{tikzcd}$$

So $\mathbb{RP}^2$ is $\mathbb{Z}/2$ orientable.

More generally, orientable := $\mathbb{Z}$-orientable $\Rightarrow$ R-orientable $\forall R$. (Given $\mathbb{Z}$-or^n $\{\mu_x\}_{x \in M}$ get R-or^n $\{\mu_x \otimes 1\}_{x \in M}$.)

→ All manifolds are $\mathbb{Z}/2$ orientable.

In general,

$$\begin{tikzcd} M_1 \arrow[dr] & \subseteq & M_2 \arrow[d] \\ & & M \end{tikzcd}$$

is called the orientation double cover of M.

Two possibilities: ① it's a trivial double cover, i.e.

$$\begin{tikzcd} M_1 \cong M \sqcup M \arrow[d] \\ M \end{tikzcd}$$

True iff M is orientable, in which case $\exists$ 2 orientations for M connected.

② It's a nontrivial cover $\iff$ M is not $\mathbb{Z}$ orientable.

For a topological manifold $M = M^n$ and a ring R, an R-orientation of M is defined to be "a compatible system of generators of the free rk-1 R-modules $H_n(M|x;R)$ $\forall x \in M$"

(Equivalently, a section

$$\begin{tikzcd} & M_R \arrow[d] \\ M \arrow[ur, dashed] \arrow[r, "="'] & M \end{tikzcd}$$

covering space w/ fibers: $H_n(M|x;R) \mapsto \{x\}$

taking values in generators of each fiber.

Note: Given $R \to S$ a ring hom, R-or $\rightsquigarrow$ S-or

Thm: 3.26: M^n closed, connected. If M is R-orientable, then $\forall x \in M$,

$$H_n(M;R) \xrightarrow{\cong} H_n(M|x;R)$$

$$H_n((M,\emptyset) \hookrightarrow (M, M\setminus\{x\}), R)$$

Def: An R fundamental class of M^n is $\mu = [M] \in H_n(M;R)$ s.t. $\mu \longmapsto \mu_x \in H_n(M|x;R)$

↖ pointwise R-or of M at x.

Ex: For $M = S^n = \Delta^n_N \underset{\partial\Delta^n}{\sqcup} \Delta^n_S$, $R = \mathbb{Z}$, have fund class

$$[M] = [\Delta^n_N] - [\Delta^n_S]. \quad \left(\text{in } H_n^{\Delta}(S^n) \xrightarrow{\cong} H_n^{\text{sing}}(S^n)\right)$$

Why?

↳ For $x \in \mathring{\Delta}^n_N$ (or $\mathring{\Delta}^n_S$),

$$\begin{tikzcd} H_n(S^n|x) \arrow[r, phantom, "\cong"] & H_n(\Delta^n_N|x) \arrow[r, phantom, "\cong \cdots \cong"] & \mathbb{Z} \\ {[M]} \arrow[u, phantom, "\in" rotate=90] \arrow[r] & {[\Delta^n_N]} & \end{tikzcd}$$

↳ For $x \in \partial\Delta^n \subset S^n$ (equator), we can rewrite $[M] \in H_n(S^n)$ in such a way that $x \in$ interior of one of the n-simps.

More generally, for any M^n w/ Δ-cx str., for M orientable, every fundamental class is of the form $[M] = \sum_\alpha \underset{\pm 1}{\varepsilon_\alpha} \cdot \left[\substack{\alpha^{th} \\ n\text{-simplex}}\right]$.

Similarly, for any (possibly non-orientable) M^n w/ Δ-cx str. the unique $\mathbb{Z}/2$ fund. class is

$$[M] = \sum_\alpha \left[\substack{\alpha^{th} \\ n\text{-simp}}\right] \in H_n(M; \mathbb{Z}/2)$$

Def: $X \in \text{Top}$, R a ring, $k \geq \ell \geq 0$ the cap product is

$$C_k(X;R) \times C^\ell(X;R) \xrightarrow{\frown} C_{k-\ell}(X;R)$$

$$(\sigma, \varphi) \longmapsto \sigma \frown \varphi := \underbrace{\varphi(\sigma|_{\Delta^{\{0,\dots,\ell\}}})}_{\text{elt of } R} \cdot \underbrace{\sigma|_{\Delta^{\{\ell,\dots,k\}}}}_{\text{singular } (k-\ell) \text{ simplex in } X}$$

Lemma: This is R-bilinear, and:

$$\partial(\sigma \frown \varphi) = (-1)^\ell \cdot (\partial\sigma \frown \varphi - \sigma \frown \delta\varphi).$$

Cor: Cap product descends to (co)homology.

i.e. $H_k(X;R) \times H^\ell(X;R) \xrightarrow{\frown} H_{k-\ell}(X;R)$.

e.g. if $[x] = [x'] \in H_k$, say $x - x' = \partial y$ for $y \in C_{k+1}$, then

$$\begin{aligned} ([x]-[x']) \frown [\varphi] &:= [(x-x') \frown \varphi] \\ &= [\partial y \frown \varphi] \\ &= [(-1)^\ell \cdot (\partial y \frown \varphi - \sigma \frown \underbrace{\delta\varphi}_{=0!})] \\ &= [\partial(y \frown \varphi)] = 0 \in H_{k-\ell} \end{aligned}$$

Remark: This can be defined for <u>simplicial</u> (co)homology, and under $H_*^\Delta \cong H_*^{sing}$ & $H_\Delta^* \cong H_{sing}^*$, cap products coincide.

Thm (Poincaré Duality):

M^n is a closed (R-orientable) manifold with R-fundamental class $[M]$. Then,

$$D := [M] \frown (-) : H^i(M;R) \xrightarrow{\cong} H_{n-i}(M;R).$$

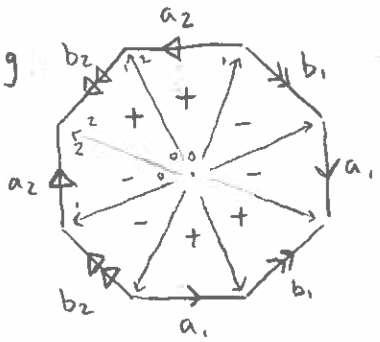
Remark: Say R a field for simplicity. Then,

$$H_i(M;R)^\vee \underset{UCT}{\cong} H^i(M;R) \underset{P.D.}{\xrightarrow{\cong}} H_{n-i}(M;R)$$

In particular, $\dim H_i = \dim H_{n-i}$

This is of the flavor of the symmetry that exists in Pascal's triangle, i.e. $\binom{n}{i} = \binom{n}{n-i}$.

Ex: $M = M_g$



(sign chosen so that touching edges cancel sign. initial choice of +/- arbitrary).

Fund class $[M] = \sum_\alpha \varepsilon_\alpha \cdot [\alpha^{th}\ \text{2-simplex}]$

$H_1(M) \cong \mathbb{Z}\{[a_i],[b_i]\}$

UCT: $H^1(M) \cong \mathbb{Z}\{[\alpha_i],[\beta_i]\}$ ← dual generator

α_i = signed inter.

<u>Summary</u>: M^n a (topological) n-mfld.

An R-orientation of M is a compatible system of R-module generators $\{\mu_x \in H_n(M|x;R)\}_{x\in M}$

($\mathbb{Z}$-or^n = or^n; always $\exists!$ $\mathbb{Z}/2$ or^n).

Corr (3.37): For any odd dimensional manifold M, $\chi(M) = 0$.

Pf: Using $R = \mathbb{Z}/2$ coefficients,

$$\hom_{\mathbb{Z}/2}(H_i(M;\mathbb{Z}/2);\mathbb{Z}/2) \overset{UCT}{\cong} H^i(M;\mathbb{Z}/2) \xrightarrow[\cong]{P.D.} H_{n-i}(M;\mathbb{Z}/2)$$

so $\dim(H_i) = \dim(H_{n-i})$.

Hence $\chi(M) = \sum_i (-1)^i \dim_{\mathbb{Z}/2}(H_i(M;\mathbb{Z}/2))$.

Since n is odd, these canonical pairs cancel

$h_0 - h_1 + h_2 - h_3 = 0$. $\blacksquare$

Corr: (3.38): Special case: $R = F$, a field. M^n a closed R-orientable n-manifold.

Then $\forall i$,

$$\begin{tikzcd}
H^i(M;R) \times H^{n-i}(M;R) \arrow[r, "\smile"] \arrow[rr, bend right, "((-)\smile(-))[M]"'] & H^n(M;R) \arrow[r, "{[M]\frown(-)}"] & R
\end{tikzcd}$$

is a <u>perfect pairing</u>.

i.e. $V \times W \longrightarrow F$

s.t. $V \longrightarrow W^\vee := \hom_F(W,F)$

and $W \rightarrow V^\vee$

are isos. Similarly, for $R = \mathbb{Z}$, but now $\frac{H^i}{\text{torsion}}$.

Main input (given PD): $\psi(\alpha \frown \varphi) = (\varphi \smile \psi)(\alpha)$.

Ex: Recall $H_i(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z} & 0 \le i \le 2n \text{ even} \\ 0 & \text{o.w.} \end{cases}$

By UCT, $H^i(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z} & 0 \le i \le 2n \text{ even} \\ 0 & \text{o.w.} \end{cases}$

Q: What is the <u>ring structure</u> on $H^*(\mathbb{C}P^n) := \bigoplus_{i\ge 0} H^i(\mathbb{C}P^n)$? (w/ cup product)

A: $H^*(\mathbb{C}P^n) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$ for $\deg(\alpha) = 2$.

Pf by induction + Cor 3.38. Clearly true for $n=0$ ($\mathbb{C}P^0 = pt$) & $n=1$ ($\mathbb{C}P^1 \cong S^2$).

Have $\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$, which gives us

$$H^*(\mathbb{C}P^{n-1}) \longleftarrow H^*(\mathbb{C}P^n)$$

(a ring map because cup products are natural). is a levelwise iso of graded abelian groups for dimensions $< 2n$. So by induction, it suffices to show that $(\alpha \cup \alpha^{n-1}) \in H^{2n}(\mathbb{C}P^n)$ is a <u>generator</u>. By perfectness of the pairing $H^2 \times H^{2n-2} \xrightarrow{\smile} H^{2n}(\mathbb{C}P^n) \xrightarrow{(-)[\mathbb{C}P^n]} \mathbb{Z}$ since $\alpha \in H^2$ is a generator, there must exist $\beta \in H^{2n-2}$ s.t. $(\alpha \cup \beta)([\mathbb{C}P^n]) = 1$, i.e. $\alpha \cup \beta$ is a generator. Now, $\alpha^{n-1} \in H^{2n-2}(\mathbb{C}P^n)$ is a generator, so $\exists! \, k \in \mathbb{Z}$ s.t. $k \cdot \alpha^{n-1} = \beta$. So, $\alpha \cup \beta = \alpha \cup (k \cdot \alpha^{n-1}) = k \cdot (\alpha \cup \alpha^{n-1})$; since $\alpha \cup \beta \in H^{2n}$ is a generator, it must be that $k = \pm 1$. So, $\alpha \cup \alpha^{n-1} \in H^{2n}$ is also a generator. ■

Similarly, $H^*(\mathbb{R}P^n; \mathbb{Z}/2) \overset{\text{as rings}}{\cong} (\mathbb{Z}/2)[\gamma]/\gamma^{n+1}$ for $\deg(\gamma) = 1$.